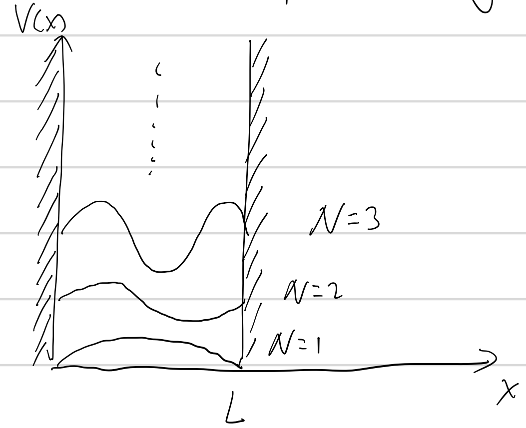
Let's keep working on Quantum well, but extend our analysis from eigenstates to general states.




Eigenstates:

$$\psi_n(x,t) = \sqrt{\frac{2}{L}}\,\sin\left(\frac{N\pi x}{L}\right) e^{-iN^2\omega_0 t} \;;$$

$$E_N = \frac{\pi^2\hbar^2}{2mL^2}N^2 \;; \quad \omega_0 = \frac{E_{N=1}}{\hbar} = \frac{\pi^2\hbar}{2mL^2}.$$

In general, one can express a state of quantum well as:

$$\psi(x,t) = \sum_{n=1}^{\infty} \alpha_n \psi_n(x,t) \;; \qquad \psi(x,t) \text{ satisfy Schrodinger equation.}$$

Normalization : $\int_0^L |\psi(x,t)|^2 dx = 1$

$$= \int_0^L \sum_{n=1}^{\infty} \alpha_n^* \psi_n^*(x,t) \cdot \sum_{m=1}^{\infty} \alpha_m \psi_m(x,t)\, dx$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \alpha_n^* \alpha_m \int_0^L \psi_n^*(x,t)\,\psi_m(x,t)\, dx.$$

We know that $\psi_n(x,t)$ is normalized, so when $n=m$, $\int_0^L \to \int_0^L |\psi_n|^2 dx = 1$

when $n \neq m$, $\int_0^L \to \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$

Use: $\sin\alpha \sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

$$\int = \frac{1}{2}\int_0^L \left( \cos\left[\frac{(n-m)\pi x}{L}\right] - \cos\left[\frac{(n+m)\pi x}{L}\right] \right) dx$$

when $n-m \neq 0$, $n+m \neq 0$, $\int_0^L \cos = 0$.

$\therefore \int = 0$ when $n \neq m$.

$\int_0^L \psi_n^* \psi_m \, dx = \delta_{n,m}$ : True for all eigenfunctions.

$$\therefore \quad 1 = \int_0^L |\psi|^2 dx = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \alpha_n^* \alpha_m \cdot \delta_{m,n} = \sum_{n=1}^{\infty} |\alpha_n|^2 = 1.$$

Normalization condition: $\sum_{n=1}^{\infty} |\alpha_n|^2 = 1$.

Now let's work on the simplest non-eigenstate:
a superposition of ground state and first excited state:

$$\psi_{1,2}(x,t) = \frac{1}{\sqrt{2}}\psi_1(x,t) + \frac{1}{\sqrt{2}}\psi_2(x,t)$$

$$= \sqrt{\frac{1}{L}} \sin\left(\frac{\pi x}{L}\right) e^{-i\omega_0 t} + \sqrt{\frac{1}{L}} \sin\left(\frac{2\pi x}{L}\right) e^{-4i\omega_0 t}$$

And we ask ourselves with the same question, what is
$\langle x \rangle$, $\langle x^2 \rangle$, $\langle p \rangle$, $\langle p^2 \rangle$ ?

1. position.

Individually, $\Psi_1(x,t)$ has average position at $\langle x \rangle_1 = \frac{L}{2}$

$\Psi_2(x,t)$ also has averge position at $\langle x \rangle_2 = \frac{L}{2}$.

Can you predict $\langle x \rangle_{1,2}$ ? Is it $\frac{L}{2}$ ?

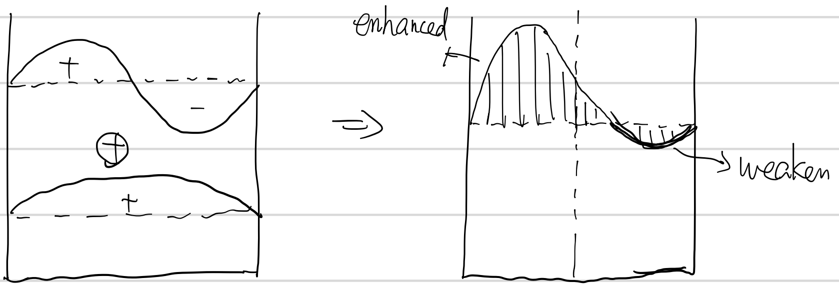
Actually, no. It is not.

So this is like the box experiments in the first class. You think in a classical way, you get the wrong answer.

Let's look at $t=0$

$$\Psi_{1,2}(x, t=0) = \sqrt{\frac{1}{L}}\left(\sin\frac{\pi x}{L} + \sin\frac{2\pi x}{L}\right)$$

Can you plot the wave function?




Interference effect.

Similar to Young's double split experiment.

$\langle x \rangle_{1,2}$ should be smaller than $\frac{L}{2}$ at $t=0$.

So it's at the left side of the potential well.

But how is that possible? Our potential well is symmetric.

There will be a conflict here if the particle prefer to stay at one side of the potential well.

So there's only one possible way to resolve this conflict.

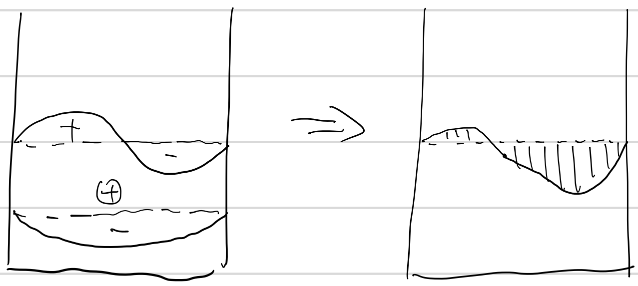
That the particle is moving. It will move from left to right, right to left, and on average, it spends same amount of time on the left and right.

Let's see if this is possible.

At $t = t' = \frac{\pi}{\omega_0}$,

$$\psi(x,t) = \sqrt{\frac{1}{L}} \sin\frac{\pi x}{L} e^{-i\omega_0 t} + \sqrt{\frac{1}{L}} \sin\frac{2\pi x}{L} e^{-4i\omega_0 t}$$

$$\Rightarrow \psi(x,t)\Big|_{t=t'} = -\sqrt{\frac{1}{L}} \sin\frac{\pi x}{L} + \sqrt{\frac{1}{L}} \sin\frac{2\pi x}{L}$$



As the probability is $|\psi|^2$,

So now the particle moves to the right side of the potential well.

So $\langle x\rangle_{1,2}$ is a function of time!

$$\langle x\rangle_{1,2} = \int_0^L \psi^*(x,t)\, x\, \psi(x,t)\, dx$$

$$= \frac{1}{L}\cdot\int_0^L \left(\sin\frac{\pi x}{L}\, e^{+i\omega_0 t} + \sin\frac{2\pi x}{L}\, e^{4i\omega_0 t}\right) x \left(\sin\frac{\pi x}{L}\, e^{-i\omega_0 t} + \sin\frac{2\pi x}{L}\, e^{-4i\omega_0 t}\right) dx.$$

$$= \frac{1}{L}\int_0^L \Big\{ \underbrace{x\sin^2\frac{\pi x}{L}}_{\sim \frac{\langle x\rangle_1}{2}} + \underbrace{x\sin^2\frac{2\pi x}{L}}_{\sim \frac{\langle x\rangle_2}{2}} + x\sin\frac{\pi x}{L}\sin\frac{2\pi x}{L}\left(e^{-3i\omega_0 t} + e^{3i\omega_0 t}\right) \Big\}\, dx$$

$$= \frac{L}{2} + \frac{1}{L}\int_0^L 2x\sin\frac{\pi x}{L}\sin\frac{2\pi x}{L}\cos 3\omega_0 t\, dx$$

$$= \frac{L}{2} + \frac{2\cos 3\omega_0 t}{L}\int_0^L x\sin\frac{\pi x}{L}\sin\frac{2\pi x}{L}\, dx$$

Use: $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

$$= \frac{L}{2} + \frac{\cos 3\omega_0 t}{L}\left[\int_0^L x\cos\frac{\pi x}{L}\, dx - \int_0^L x\cos\frac{3\pi x}{L}\, dx\right]$$

$$\frac{L}{\pi}\int_0^L x\, d\sin\frac{\pi x}{L}$$

$$= \frac{L}{\pi} x\sin\frac{\pi x}{L}\Big|_0^L - \frac{L}{\pi}\int \sin\frac{\pi x}{L}\, dx$$

$$= +\frac{L^2}{\pi^2}\cos\frac{\pi x}{L}\Big|_0^L = \frac{L^2}{\pi^2}(\cos\pi - \cos 0)$$

$$= \frac{-2L^2}{\pi^2}$$

$$-\int_0^L \frac{L}{3\pi}\, x\, d\sin\frac{3\pi x}{L}$$

$$= \frac{-L}{3\pi} x\sin\frac{3\pi x}{L}\Big|_0^L + \int_0^L \frac{L}{3\pi}\sin\frac{3\pi x}{L}\cdot dx$$

$$= -\frac{L^2}{(3\pi)^2}\cos\frac{3\pi x}{L}\Big|_0^L = \frac{2L^2}{(3\pi)^2}$$

$\therefore \quad \langle x \rangle_{1,2} = \frac{L}{2} + \cos 3\omega_0 t \left( -\frac{2L}{\lambda^2} + \frac{2L}{\lambda^2} \frac{1}{9} \right)$

$= \frac{L}{2} - \frac{16L}{9\lambda^2} \cos 3\omega_0 t.$

Verify: at $t = 0$; $\langle x \rangle_{1,2} = \frac{L}{2} - \frac{16L}{9\lambda^2} < \frac{L}{2}$ ✓

at $t = \frac{\lambda}{\omega_0}$; $\langle x \rangle_{1,2} = \frac{L}{2} + \frac{16L}{9\lambda^2} > \frac{L}{2}$ ✓.

Take away: Interference matters!

Question: since $\langle x \rangle_{1,2}$ is function of time:
can we define velocity as $\frac{d}{dt}\langle x \rangle_{1,2}$ ?

And then $\langle \hat{p} \rangle = m \cdot \frac{d\langle x \rangle}{dt}$ ?

Is it possible?

Let's do some analysis: $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

$$\psi(x,t) = \sqrt{\frac{1}{L}} \left\{ \frac{e^{\frac{i\lambda x}{L} - i\omega_0 t} - e^{-\frac{i\lambda x}{L} - i\omega_0 t}}{2i} + \frac{e^{\frac{i2\lambda}{L}x - i4\omega_0 t} - e^{-\frac{2i\lambda x}{L} - 4i\omega_0 t}}{2i} \right\}.$$

$\psi(x,t)$ is composed of four eigenstates, $\pm \hbar k_0$, $\pm 2\hbar k_0$.

According to postulate: $\psi = \sum_{n=1}^{\infty} \alpha_{n,p} \psi_{n,p}$

$\langle \hat{p} \rangle = \sum_n |\alpha_{n,p}|^2 \cdot p_n. = \frac{1}{4} (\hbar k_0 - \hbar k_0 + 2\hbar k_0 - 2\hbar k_0)$

$= 0. \qquad ; k_0 = \frac{\pi}{L}$

Is this correct ?

Actually no. The condition for this postulate is that

$\psi_n$ is orthonormal.

$$\int_0^L \psi_{p,n}^* \psi_{p,m} \, dx \propto \int_0^L e^{-ink_0x} \cdot e^{imk_0x} \, dx = \int_0^L e^{i(m-n)\frac{\pi}{L}x} \, dx.$$

When $m-n$ is odd number, the integration is not zero.

$$\int_0^L e^{\frac{i\pi x}{L}} dx = \int_0^L \cos\frac{\pi x}{L} dx + i\int_0^L \sin\frac{\pi x}{L} dx$$

$$= \underbrace{\frac{L}{\pi} \sin\frac{\pi x}{L}\Big|_0^L}_{=0} - \underbrace{\frac{iL}{\pi} \cos\frac{\pi x}{L}\Big|_0^L}_{=\frac{2iL}{\pi}}.$$

$\psi_n$ is not orthonormal in $(0, L)$

What about previous lecture ? That's just a happy incident

where $n = -m$, so $m - n = 2m$,

it's an even number, so the $\int \psi_m^* \cdot \psi_n \, dx = 0$ when $m = -n$.

Verify $\langle \hat{p} \rangle = \int_0^L \psi^* \hat{p} \psi \, dx$

$$= \frac{\hbar}{iL} \int_0^L \left( \sin\frac{\pi x}{L} e^{+i\omega_0 t} + \sin\frac{2\pi x}{L} e^{+i4\omega_0 t} \right) \frac{\partial}{\partial x} \left( \sin\frac{\pi x}{L} e^{-i\omega_0 t} + \sin\frac{2\pi x}{L} e^{-i4\omega_0 t} \right) dx$$

$$= \frac{\hbar}{iL} \int_0^L (\quad) \left( \frac{\pi}{L} \cos\frac{\pi x}{L} e^{-i\omega_0 t} + \frac{2\pi}{L} \cos\frac{2\pi x}{L} e^{-i4\omega_0 t} \right) dx$$

$$= \frac{\hbar}{iL} \int_0^L \Big\{ \frac{\pi}{L} \sin\frac{\pi x}{L} \cos\frac{\pi x}{L} + \frac{2\pi}{L} \sin\frac{\pi x}{L} \cos\frac{2\pi x}{L} e^{-i3\omega_0 t}$$
$$+ \frac{\pi}{L} \sin\frac{2\pi x}{L} \cos\frac{\pi x}{L} e^{i3\omega_0 t} + \frac{2\pi}{L} \sin\frac{2\pi x}{L} \cos\frac{2\pi x}{L} \Big\} dx$$

Use: $\sin\alpha \cos\beta = \frac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{2}$

$$= \frac{\hbar}{iL} \int_0^L \frac{\pi}{2L} \sin\frac{2\pi x}{L} + \frac{\pi}{L} \sin\frac{3\pi x}{L} e^{-i3\omega_0 t} - \frac{\pi}{L} \sin\frac{\pi x}{L} e^{-i3\omega_0 t}$$
$$+ \frac{\pi}{2L} \sin\frac{3\pi x}{L} e^{i3\omega_0 t} + \frac{\pi}{2L} \sin\frac{\pi x}{L} e^{+i3\omega_0 t} + \frac{\pi}{L} \sin\left(\frac{4\pi x}{L}\right) \Big\} dx.$$

We know: $\int_0^L \sin\frac{\pi x}{L} dx = \frac{-L}{\pi} \cos\frac{\pi x}{L}\Big|_0^L = \frac{-L}{\pi}(-1-1) = \frac{2L}{\pi}.$

$$\int_0^L \sin\frac{2\pi x}{L} dx = \frac{-L}{2\pi} \cos\frac{2\pi x}{L}\Big|_0^L = 0$$

$$\int_0^L \sin\frac{3\pi x}{L} dx = \frac{-L}{3\pi} \cos\frac{3\pi x}{L}\Big|_0^L = \frac{-L}{3\pi}(-1-1) = \frac{2L}{3\pi}$$

$$\int_0^L \sin\frac{4\pi x}{L} dx = \frac{-L}{4\pi} \cos\frac{4\pi x}{L}\Big|_0^L = 0.$$

$$\therefore \langle \hat{p} \rangle = \frac{\hbar}{iL} \left\{ \frac{2}{3} e^{-i3\omega_0 t} - 2e^{-i3\omega_0 t} + \frac{1}{3} e^{i3\omega_0 t} + e^{+i3\omega_0 t} \right\}$$

$$= \frac{4\hbar}{3iL} \left\{ \frac{e^{i3\omega_0 t} - e^{-i3\omega_0 t}}{2i} \right\} \times 2i$$

$$= \frac{8\hbar}{3L} \sin(3\omega_0 t).$$

$$\langle \hat{x} \rangle = \frac{L}{2} - \frac{16L}{9\pi^2} \cos(3\omega_0 t)$$

$$m \cdot \frac{d\langle \hat{x} \rangle}{dt} = \frac{+16mL}{9\pi^2} \times 3\omega_0 \sin(3\omega_0 t) = \frac{16mL \cdot \omega_0}{3\pi^2} \sin(3\omega_0 t)$$

Use: $\omega_0 = \frac{E_{n=1}}{\hbar} = \frac{\pi^2 \hbar}{2mL^2}.$

$$m \cdot \frac{d\langle \hat{x} \rangle}{dt} = \frac{16mL}{3\pi^2} \cdot \frac{\pi^2 \cdot \hbar}{2mL^2} \cdot \sin(3\omega_0 t) = \frac{8\hbar}{3L} \sin(3\omega_0 t).$$

Interesting: $m \cdot \frac{d\langle \hat{x} \rangle}{dt} = \langle \hat{p} \rangle.$

The classical definition of momentum works when taking average values in quantum system.

Later in this class, we will show this is a simple version of Heisenberg equation of motion.

Additional reading:

If the intitial condition is $\psi(x, t=0) = f(x)$,

then the way to solve $\psi(x,t)$, $\langle x \rangle$, $\langle p \rangle$ is:

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \psi_n(x,t)\Big|_{t=0}.$$

$$\Rightarrow \int_0^L \psi_m^* f(x)\,dx = \sum_{n=1}^{\infty} \int_0^L \alpha_n \cdot \psi_m^* \psi_n\,dx$$

Use orthornormal of $\psi_m$, $\psi_n$,

$$\int_0^L \psi_m^* f(x)\,dx = \sum_{n=1}^{\infty} \alpha_n \cdot \delta_{m,n}$$

$$\Rightarrow \alpha_n = \int_0^L \psi_m^*(x, t=0) f(x)\,dx.$$

Use this method, you can find all the coefficient, and then use the previous method for superposition states to calculate $\langle x \rangle$, $\langle p \rangle$ at time $t$.

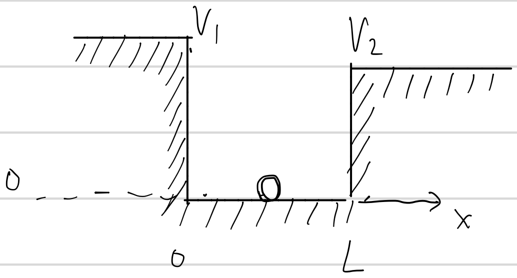
Example: $f(x) = \delta(x - \frac{L}{2})$ put electron at the center.
Not normalized.

Notice: $f(x)$ must satisfy B.C. $f(x=0) = f(x=L) = 0$.
Otherwise it is not a valid wave-equation.

More on solving Schordinger equation:

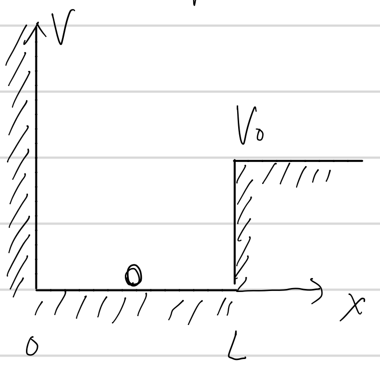
Real quantum well & Continuous wavefunction condition

We know that in reality, there is no inifity deep potential well, maybe except black hole...




A real quantum well usually looks like this: two uneven potential.

There are special tricks to solve this type of problem, and for simplicity, we just let $V_1 \to \infty$, $V_2 = V_0$.




One side infinite potential well.

We want to solve $\psi(x)$ so that we can know the behavior of the electron.

Schordinger equation:

$$\left\{ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \right\} \psi(x,t) = i\hbar \frac{\partial}{\partial t} \psi(x,t)$$

$$\Rightarrow \left\{ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \right\} \psi_n(x) = E_n \psi_n(x)$$

$$V(x) = \begin{cases} \infty, & x<0 \\ 0, & 0<x<L \\ V_0, & x>L. \end{cases}$$

Intuitively, or from classical physics, we know that if $E_n > V_0$, the electron will have enough energy to escape from the potential well.
So there exists two solutions, $\psi_n(x)$ at $E_n < V_0$; $\psi_m(x)$ at $E_m > V_0$.

We will show this in the calculation.

For each of these three regimes, we know the solution to time-independent Schordinger equation.

1°. $x<0$, $V \to \infty$, $\psi_n(x) = 0$, at $x<0$.

2°. $0 < x < L$, $V = 0$;

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_n(x) = E_n \psi_n(x)$$

$$\psi_n(x) = \sin\sqrt{\frac{2mE_n}{\hbar^2}}\,x \quad \text{or} \quad \cos\sqrt{\frac{2mE_n}{\hbar^2}}\,x$$

3°. at $x > L$, $V = V_0$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_n(x) + V_0\psi_n(x) = E_n\psi_n(x)$$

$$\Rightarrow \quad -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_n(x) = (E_n - V_0)\,\psi_n(x).$$

Notice: for $E_n - V_0 > 0$, the solution is similar:

$$\psi_n(x) = \sin\sqrt{\frac{2m(E_n-V_0)}{\hbar^2}}\,x \quad \text{or} \quad \cos\sqrt{\frac{2m(E_n-V_0)}{\hbar^2}}\,x$$

But for $E_n - V_0 < 0$ $\quad \sqrt{-\dots}$ gives complex number $\sqrt{-1} = i$

We can cheek :

$$\frac{\partial^2 \psi_n(x)}{\partial x^2} = \frac{2m(V_0 - E_n)}{\hbar^2} \psi_n(x)$$

$\hookrightarrow > 0.$

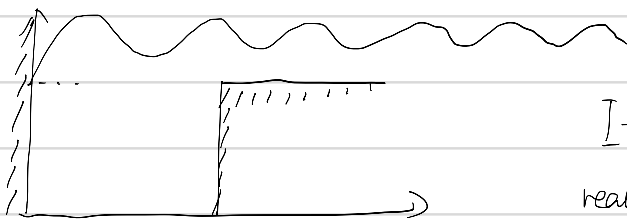
The solution to this is:

$$\psi_n(x) = e^{-\sqrt{\frac{2m(V_0-E_n)}{\hbar^2}}\, x} \quad , \text{ or } e^{\sqrt{\frac{2m(V_0-E_n)}{\hbar^2}}\, x} .$$

Both cases are interesting.

Let's look at $E_n - V_0 > 0$ first.

The solution at $x > L$ is travelling waves. So electron can escape:



It's an interesting solution but we don't really care about it, as you lose your electron ...

We will focus on $E_n - V_0 < 0$ .

Introduce: continous condition for wavefunction.

$\psi(x)$ must be continous, otherwise $\frac{\partial \psi}{\partial x} \to \infty$.

$\frac{\partial \psi(x)}{\partial x}$ must be continous, otherwise $\frac{\partial^2 \psi}{\partial x^2} \to \infty$. ( exception: $\delta$-function potential well )

$$-\frac{\hbar^2}{2m}\underbrace{\frac{\partial^2 \psi}{\partial x^2}}_{\text{must be finite}} + \underbrace{V(x)\,\psi(x)}_{\text{finite}} = \underbrace{E_n\,\psi(x)}_{\text{finite}}$$

Solutions in three regime is connected by continous condition.

$$\psi_n(x) = \begin{cases} 0 & x<0 \\ a\sin\sqrt{\frac{2mE_n}{\hbar^2}}\,x + b\cos\sqrt{\frac{2mE_n}{\hbar^2}}\,x\ ; & 0<x<L \\ c\,e^{-\sqrt{\frac{2m(V_0-E_n)}{\hbar^2}}\,x} + d\,e^{\sqrt{\frac{2m(V_0-E_n)}{\hbar^2}}\,x}\ ; & x>L. \end{cases}$$

B.C. we know the electron is confined, so $|\psi_n(x)|^2 \to 0$ at $x\to\infty$.
$\therefore\ d=0$.

Continous condition: $\psi_n(x=0)=0$ ; so $b=0$.

$$\psi_n(x) = \begin{cases} 0 & x<0 \\ a\sin\sqrt{\frac{2mE_n}{\hbar^2}}\,x & 0<x<L \\ c\,e^{-\sqrt{\frac{2m(V_0-E_n)}{\hbar^2}}\,x} & x>L. \end{cases}$$

Unknow parameters: $a$, $c$, $E_n$. Need 3 equations.

From homeworks, you will show $\psi_n(x)$ can always be real, so we pick $a$, $c$ to be real number.

Continuous equation 1°: $\psi_n(x=L_-) = \psi_n(x=L_+)$

$$a \sin\sqrt{\frac{2mE_n}{\hbar^2}}\,L = c\, e^{-\sqrt{\frac{2m(V_0-E_n)}{\hbar^2}}\,L} \quad ①$$

2°. $\frac{\partial \psi_n}{\partial x}(x=L_-) = \frac{\partial \psi_n}{\partial x}(x=L_+)$

$$\Rightarrow\ a\cdot\sqrt{\frac{2mE_n}{\hbar^2}}\cos\sqrt{\frac{2mE_n}{\hbar^2}}\,L = -c\sqrt{\frac{2m(V_0-E_n)}{\hbar^2}}\cdot e^{-\sqrt{\frac{2m(V_0-E_n)}{\hbar^2}}\,L} \quad ②$$

3°. Normalization of probability:

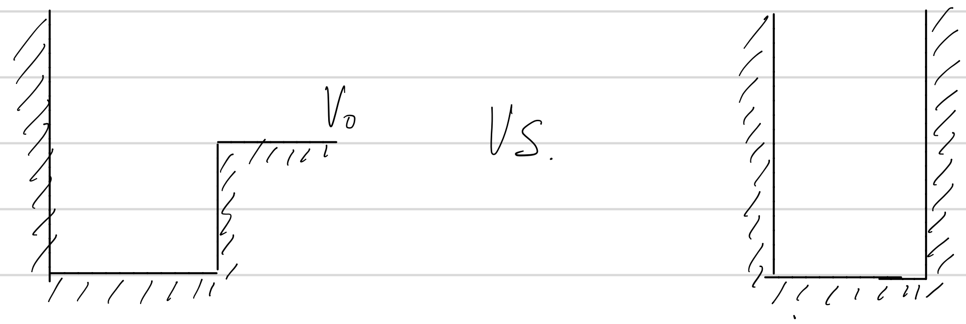
$$\int_0^\infty |\psi_n(x)|^2 dx = 1$$

$$\Rightarrow \int_0^L a^2 \sin^2\sqrt{\frac{2mE_n}{\hbar^2}}\,x\,dx + \int_L^\infty c^2 e^{-2\sqrt{\frac{2m(V_0-E_n)}{\hbar^2}}\,x}\,dx = 1 \quad ③.$$

We care mostly about $E_n$, the energy.

We can do ① / ②

But let's first do a physics analysis.




Which has higher energy for $E_n$? $(E_n < V_0)$

Take an extreme case, for $n=1$, ground state,
if $V_0 \to 0$, it's half open. So the ground state will have infinitely long wavelength (to keep its energy low).

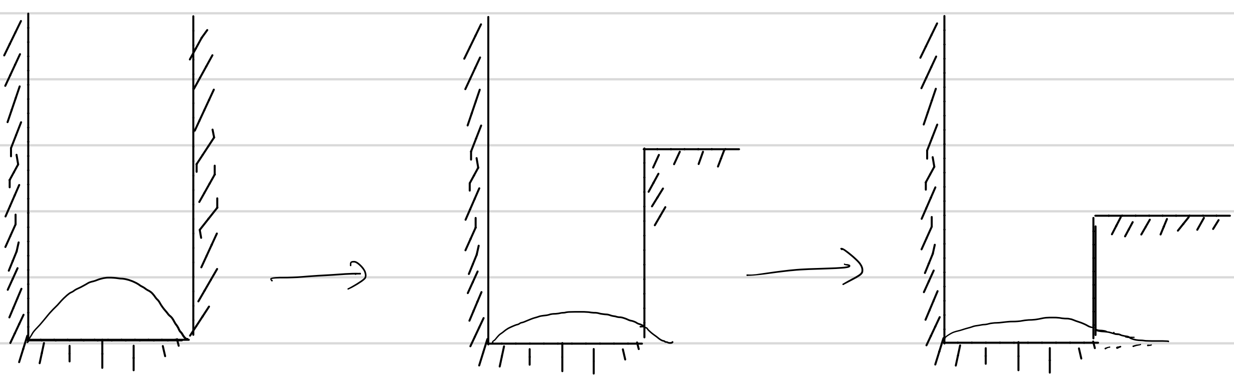
$E_{n=1} \to 0$.

When $V_0 \to \infty$, it evolves into infinite potential well.
And $E_{n=1} \not\to 0$.

In general, $E_n(V_0)$ will change continously with parameter $V_0$, so we can guess:

$E_n(V_0) < E_n$, as long as $E_n < V_0$.

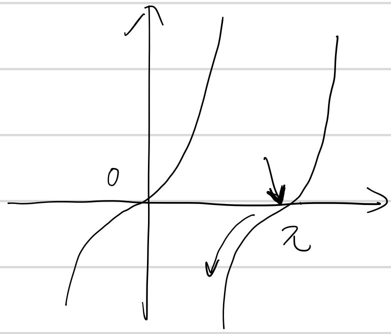
As $E_n$ is smaller, the electron should have longer wavelength.



Use ① / ②:

$$\frac{\sin\sqrt{\frac{2mE_n}{\hbar^2}}L}{\sqrt{\frac{2mE_n}{\hbar^2}}\cos\sqrt{\frac{2mE_n}{\hbar^2}}L} = -\frac{1}{\sqrt{\frac{2m(V_0-E_n)}{\hbar^2}}}$$

$$\Rightarrow \tan\sqrt{\frac{2mE_n}{\hbar^2}}L = -\sqrt{\frac{E_n}{(V_0-E_n)}}$$




Difficult to solve...

At $V_0 \to \infty$, $\tan\sqrt{\frac{2mE_n}{\hbar^2}}L \to -0$.

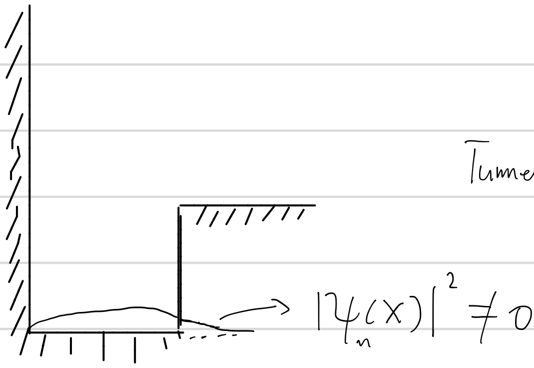
$\therefore \sqrt{\frac{2mE_n}{\hbar^2}}L = N\pi$

$E_n = \frac{N^2\pi^2\hbar^2}{2mL^2}$ ; same as previous.

When $V_0$ decrease from infinity, right-hand side decreases

So: $\sqrt{\frac{2mE_n}{\hbar^2}}L \downarrow$ ; $E_n \downarrow$.

Tunneling:




$$e^{-\sqrt{\frac{2m(V_0-E_n)}{\hbar^2}}\,x}$$

Tunneling depth: $\sqrt{\frac{2m(V_0-E_n)}{\hbar^2}}\,x_0 = 1$

$$x_0 = \frac{\hbar}{\sqrt{2m(V_0-E_n)}}.$$

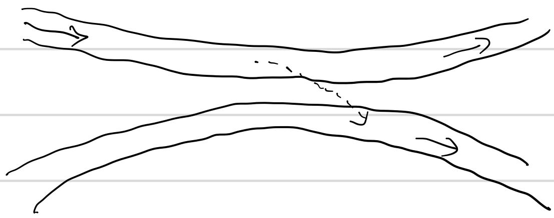
Classically, if $E_n < V_0$, there is zero probability of finding the electron outside the potential well.

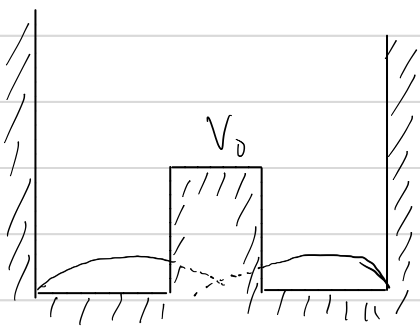
However, in quantum mechanics, this probability is no longer zero.

You can find electrons outside the potential well.

But this is actually not magic, all waves call do tunneling

In EM-wave, it is called evanescant wave. It's widely used in fiber optics. We have tens of evanescant wave coupler in our lab.






You can tunnel from one side to another, even with $E_n < V_0$.