Addition to Postulate 7 (general claim)

Given a wavefunction $\psi = \alpha\psi_1 + \beta\psi_2$, ψ_1 and ψ_2 perfectly normalized, and $\int_{-\infty}^{+\infty} \psi_1^* \psi_2 \, dx = 0$ (Orthornormal)

The probability of the system is in ψ_1 is $|\alpha|^2$.

You can derive Postulate 2 from this one...

Note 1: why need orthognal?

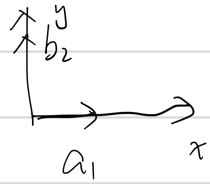
If $\psi_1(x,t) = e^{ik_1x - i\omega t}$; $\psi_2 = \frac{e^{ik_1x - i\omega t} + e^{ik_2x - i\omega t}}{\sqrt{2}}$

$\psi_2(x,t)$ contains components from ψ_1, then the probability of the system in $\psi_1(x,t) = e^{ik_1x - i\omega t}$ is not $|\alpha|^2$ anymore,

it becomes $|\alpha + \frac{\beta}{\sqrt{2}}|^2$.

Similar rules in vectors: (x,y) denote a vector.

$C = (a_1, 0) + (a_2, b_2)$; We can't say the value of x in vector c is a_1, unless $a_2 = 0$, which make these two vector perpendicular



$\int_{-\infty}^{+\infty} \psi_1^* \psi_2 \, dx = 0$ is the mathematical way of saying two functions are perpendicular to each other.

↓

Inner Product

The definition of inner product comes from linear algebra.

Inner product of $\vec{a}\cdot\vec{b} = (x_1, y_1, z_1)\cdot(x_2, y_2, z_2)$
$= x_1x_2 + y_1y_2 + z_1z_2$

If x, y, z can be complex number, then $\vec{a}^{*}\cdot\vec{b}$

You can expand this into continous function:

$$\text{In Prod}\left(\psi_1^*(x);\ \psi_2(x)\right) = \psi_1^*(x_1)\cdot\psi_2(x_1) + \psi_1^*(x_2)\cdot\psi_2(x_2) + \psi_1^*(x_3)\cdot\psi_2(x_3) + \ldots$$
$$\to \int_{-\infty}^{+\infty} \psi_1^*\psi_2\,dx.$$

This is ofter called Hilbert Space

Note 2: How do you do the measurement? Wave-function is not something you can measure.

Again $\psi = \alpha e^{ik_1x - i\omega t} + \beta e^{ik_2x - i\omega t} = \alpha\psi_1 + \beta\psi_2$

We can measure momentum of the system. If the momentum is $\hbar k_1$, then correspond to ψ_1

Later we will talk more about measurement.

Operator and Schordinger equation.

Let's take a closer look at plane wave, as the superposition of plane waves can describe any wave function.

This time we also include "time".

$$\psi(x,t) = e^{ikx - i\omega t}$$

For this plane wave, we know its momentum is

$$p = \hbar k = \frac{h}{\lambda} ;$$

We know the energy is
$E = \hbar\omega$.
We also know the energy is:
$E = \frac{p^2}{2m} + V(x)$; $V(x) = 0$ for simplicity......

Now to obtain the momentum of the plane wave, we can look into ikx in the waveform, but we can also apply an operation of the plane wave:

$$\underbrace{\frac{\hbar}{i}\frac{\partial}{\partial x}}_{\text{operator}}\psi(x,t) = \frac{\hbar}{i}\frac{\partial}{\partial x} e^{ikx - i\omega t} = \hbar k e^{ikx - i\omega t} = \underbrace{p}_{\text{number}}\psi(x,t)$$

1. We call $\frac{\hbar}{i}\frac{\partial}{\partial x} = \hat{p}$ is the momentum operator.

☆ Definition: If $\hat{p}\,\psi(x,t) = p\,\psi(x,t)$, where p is a number,

$\psi(x,t)$ is an eigenstate of momentum operator, and p is an eigenvalue of momentum operator

The name, "eigen" comes from linear algebra, where if

$M \cdot X = \lambda X$, M is matrix, X is vector, λ is a number, then X is an eigenvector of M.

☆ Physically, what does eigenstate mean?

The eigenstate of $\hat{p}$, has a well-defined momentum.
Every time you measure the momentum of this eigenstate, your measurement value will be the eigenvalue, 100%.

So: $\psi_1(x,t) = e^{ik_1x - i\omega t}$ is eigenstate, momentum: $\hbar k_1$

$$\psi(x,t) = \alpha\psi_1(x,t) + \beta\psi_2(x,t) = \alpha e^{ik_1x-i\omega t} + \beta e^{ik_2x-i\omega t}$$

is not an eigenstate of $\hat{p}$.

$$\hat{p}\,\psi(x,t) = \alpha\hbar k_1 e^{ik_1x-i\omega t} + \beta\hbar k_2 e^{ik_2x-i\omega t}$$

$$= \hbar\left(k_1 \cdot \alpha e^{ik_1x-i\omega t} + k_2 \cdot \beta e^{ik_2x-i\omega t}\right)$$

$$\neq \underset{\text{number}}{p}\left(\alpha e^{ik_1x-i\omega t} + \beta e^{ik_2x-i\omega t}\right)$$

$$\neq p\,\psi(x,t).$$

It makes sense as $\psi(x,t) = \alpha \psi_1(x,t) + \beta \psi_2(x,t)$

has some uncertainty in momentum. $\hbar k_1$ or $\hbar k_2$.

✱ In general: $\hat{A}$ is an operator, α is a number,
$\psi(x,t)$ is a wave-function,

If $\hat{A}\psi(x,t) = \alpha\psi(x,t)$, we say

$\psi(x,t)$ is the eigenstate of $\hat{A}$, α is the eigenvalue of $\hat{A}$.

2. position operator: it turns out position operator is just $\hat{x} = x$.
And its eigenstate is δ-function!

(We are not digging into this, it is rather complicated in math).

3. Next thing we want to get from a plane wave:

Energy: $\hbar\omega$;

$\psi(x,t) = e^{ikx-i\omega t}$: Follow the same idea, we have:

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = i\hbar\frac{\partial}{\partial t}\left(e^{ikx-i\omega t}\right) = i\hbar(-i\omega)e^{ikx-i\omega t} = \hbar\omega e^{ikx-i\omega t}$$
$$= E\cdot e^{ikx-i\omega t}.$$

It looks like: $i\hbar\frac{\partial}{\partial t}$ is the energy operator.

But, there's also another way to write down energy:

$$\hat{E} = \frac{\hat{p}^2}{2m} + V(\hat{x}) = \hat{H} \quad (\text{Hanmiltonian})$$

kinetic energy $\quad$ potential energy.

Interesting: How about just let energy equals to energy?

☆ $$\hat{H}\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t)$$

And this is Schordinger equation.

This equation must be right, as it's energy = energy.

But whether you can solve something useful from it is yet to be proved.

Sidenote: similarly, you can write down the quantum wave equation in relativity:

In relativity, we have:

$$E = \sqrt{m_0^2 c^4 + p^2 c^2} \quad ; \quad E^2 = m_0^2 c^4 + p^2 c^2$$

rest energy^2 $\quad$ kinetic energy^2

So: $\hat{E}^2\psi = \left(i\hbar\frac{\partial}{\partial t}\right)^2\psi \Rightarrow \left(m_0^2 c^4 - \hbar^2 c^2 \frac{\partial^2}{\partial x^2}\right)\psi = -\hbar^2\frac{\partial^2}{\partial t^2}\psi$

$$\Rightarrow \left(\frac{m_0^2 c^2}{\hbar^2} - \frac{\partial^2}{\partial x^2} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi(x,t) = 0.$$

This is Klein-Gordon equation.

3-D

$$\frac{\partial^2}{\partial x^2} \rightarrow \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Notice this is a four-dimensional equation,

Your coordinate : (x, y, z, ict)

Now let's backup a little bit and extract some general posturlate from above:

Postulate 4:

Every observable attribute of a physical system (e.g. momentum, position, angular momentum and energy...) is described by an operator that acts on the wavefunctions that describe the system.

$\downarrow$ Hermitian:

extension 4.1

For every operator, there are special states that are not changed (except by a constant factor) by the action of an operator,

$$\hat{A}\psi_a = a\psi_a.$$

ψ_a is the eigenstate and a is the eigenvalue.

Note 1: what does Hermitian mean? In physics, it means the eigenvalue of this operator must be real.

In methematics form, it means

$$\int \psi_2^* (\hat{A}\psi_1)\, dx = \int (\hat{A}\psi_2)^* \psi_1 \, dx \quad ; \ \psi_1, \psi_2 \text{ are any two functions.}$$

Another name for Hermitian operator is self-adjoint operator

This is adjoint operator of $\hat{A}$, is defined as:

$$\int (\hat{A}^{+}\psi_2)^* \psi_1 dx = \int \psi_2^* \hat{A} \psi_1 \, dx .$$

$\therefore \ \hat{A}^{+} = \hat{A}$ for Hermitian operator. (Ajoint of $\hat{A}$ is $\hat{A}$ itself)

It's rather complicated to establish the connection between the physics picture and the mathematics picture ... Need serious linear algebra.

Homework: show $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$ is Hermitian operator, where you can write $\psi_1(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \tilde{\psi}_1(k)\, e^{ikx} dk$; $\psi_2(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \tilde{\psi}_2(k)\, e^{ikx} dx$.

Note 2: Because the operator of an observable attribute is Hermitian, there exists a set of eigen-functions $\psi_{a,n}$, where $\psi_{a,n}$ are orthonormal to each other.

$$\int \psi^*_{a,n_1} \cdot \psi_{a,n_2}\, dx = \delta_{n_1,n_2} \; ; \; \delta = 1, \text{ when } n_1 = n_2$$
$$= 0, \text{ when } n_1 \neq n_2.$$

Simple proof: (no degeneracy)

$$\int \psi^*_{a,n_1} \cdot \hat{A}\, \psi_{a,n_2}\, dx = \int (\hat{A}\, \psi_{a,n_1})^* \cdot \psi_{a,n_2}\, dx$$
$$\parallel$$
$$\int \psi^*_{a,n_1}\, a_{n_2} \cdot \psi_{a,n_2}\, dx = \int a^*_{n_1}\, \psi^*_{a,n_1} \cdot \psi_{a,n_2} \cdot dx$$
$$\parallel$$
$$a_{n_2} \int \psi^*_{a,n_1}\, \psi_{a,n_2}\, dx = a_{n_1} \cdot \int \psi^*_{a,n_1}\, \psi_{a,n_2}\, dx$$

$$\Rightarrow (a_{n_2} - a_{n_1}) \int \psi^*_{a,n_1} \cdot \psi_{a,n_2}\, dx = 0$$

If $a_{n_1} \neq a_{n_2}$ (two different eigenvalue, then $\int \psi^*_{a,n_1} \cdot \psi_{a,n_2}\, dx = 0$.

If $a_{n_1} = a_{n_2}$, we then have the freedom to choose ψ_{a,n_1} and ψ_{a,n_2} such that they are orthonormal (Not Important in this class)

Example: e^{ikx} is eigen-function of $\hat{p}$.

e^{ik_1x} and e^{ik_2x} are orthonormal: $\int e^{i(k_1-k_2)x}\, dx = 0$ when $k_1 \neq k_2$.

Postulate 5:

The time evolution of a quantum system obeys Schordinger equation:

$$\hat{H}\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t);$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}); \quad \hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}.$$

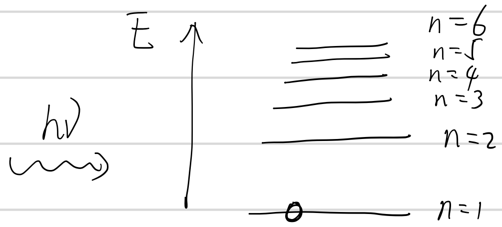
Postulate 6: The only possible result of the measurement of an observable A is one of the eigenvalues of the corresponding operator $\hat{A}$.

This is the origin of "quantum behavior". Absolutely necessary.

Example 1: In electron double-slit experiment, we are measuring observable x, position. The corresponding operator is $\hat{x}$, eigenvalue is x_0, eigenstate is $\delta(x-x_0)$. Result: individual electron is resolved.

Example 2: "Indirect" measurement:

Hydrogen atom spectrum: the photon frequency tells us the energy difference of each eigenstate. In this case, we are measuring observable energy, E, the corresponding operator is $\hat{H}$, eigenvalue is E_n (with arbitany zero point), and eigenstate is ψ_n, where $\hat{H}\psi_n = E_n\psi_n$.



Postulate 7: When measuring an observable A on state ψ, the probability of obtaining eigenvalue a_n is given by $|\alpha_n|^2$, where

$$\psi = \sum_{n=1}^{\infty} \alpha_n \psi_{a,n}$$

7.1:

The expected value when measuring an observable A on any state ψ, is

$$\langle A \rangle = \int \psi^* A \cdot \psi \, dx.$$ || Expected value:

Simple proof: $\langle A \rangle \equiv \sum_n P_n a_n$; $P_n = |\alpha_n|^2$ is the probability of a_n

As $\psi_{a,n}$ is orthornormal to each other

$$\int \psi^* (A\psi) dx = \int \sum_{n=1}^{\infty} \alpha_n^* \psi_{a,n}^* \cdot \left(A \sum_{m=1}^{\infty} \alpha_m \psi_{a,m}\right) dx$$

$$= \int \sum_{n=1}^{\infty} \alpha_n^* \psi_{a,n}^* \cdot a_m \sum_{m=1}^{\infty} \alpha_m \psi_{a,m} \, dx$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_n^* \alpha_m a_m \int \psi_{a,n}^* \cdot \psi_{a,m} \, dx$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha_n^* \alpha_m a_m \delta_{m,n} = \sum_{n=1}^{\infty} |\alpha_n|^2 \cdot a_n.$$

This is a very useful equation.

$\boxed{\text{Homework: calculate momentum for certain wavefunction.}}$

Extension of 7.2

Immediately after the measurement of A with a result of a_n, the state of the system becomes ψ_{a_n}.

Comment: This is necessary, as this is the only way that when you repeatly measure one object, it gives you the same measurement result.

Note: Postulate 7 and postulate 2 can actually be combined.

☆ These are all the postulates we need for quantum mechanics. It's your job to get super comfortable with them through homework practise, such that you don't have to think about them when you use them (just like $F = ma$).

☆ Simple version: wave function $\psi(x,t)$ describle the system, it follows $\hat{H}\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t)$ Schrodinger equation. One cannot measure $\psi(x,t)$, but only measure physics attribute ($x, p, E \ldots$), each attribute corresponds to a Hermitian operator $\hat{A}$. $\hat{A}$ has a set of eigen function $\hat{A}\psi_n = a_n\psi_n$;
Wave-function can be expressed as linear combinition of ψ_n

$\psi = \sum_{n=1}^{\infty} \alpha_n \cdot \psi_n$; When measure A, one can only get eigenvalue a_n, with probability of $|\alpha_n|^2$. The expected value for A is $\langle A \rangle = \int \psi^* \hat{A} \psi \, dx$.