Final exam of Optical Quantum Electronics

Dec 7th, 2018

Exam rules

1. The exam is open book (with internet access). Please cite the reference clearly in your answers.

2. All students shall submit their answers to the instructor before 5 pm, Dec 18th, 2018. You can put your answer sheet in the box outside E220 office, Thornton Hall. If you are traveling during the exam week, you can email me the digital version (scanning, high quality photos, iPad notes...) of your answers.

3. It is not allowed to discuss the exam with any person until 5pm, Dec 18th.

4. The exam time is any two days before 5pm, Dec 18th. It is OK to study lecture notes, textbooks between those two days, but you cannot work on the exam problems (Let's say the first and second days you pick are Monday and Friday, then you can study lecture notes on Tues-Thur). Please write down the dates of your exam on your answer sheet.

5. There is a bonus question of 20 points. However, your score will be capped at 100. Scores from 100 to 120 will all be counted as 100 points.

6. For students who are auditing this class: you can use the score of this exam when you enroll in the same course in the future. Your score is valid as long as I'm the instructor of optical quantum electronics. Keep a copy of your graded exam sheet in case I lose your score.

Correction

1. In equation (1) and (7), the E field should be $E(z,t) = E_1(z)e^{-i\omega_1 t + ik_1 z} + E_2(z)e^{-i\omega_2 t + ik_2 z} + E_3(z)e^{-i\omega_3 t + ik_3 z} + c.c.$.

2. In question 2, first line under equation (6), the E field should be $E(z,t) = E_1(z)e^{-i\omega_1 t + ik_1 z} + E_2(z)e^{-i\omega_2 t + ik_2 z} + c.c.$.

Problems

Question 1: Difference frequency generation (30 points)

We have a waveguide of length L. The input is a laser beam with two frequency components (ω_1, ω_2). The third frequency component, $\omega_3 = \omega_1 - \omega_2$, can be generated in the waveguide. The electrical field of the light inside the waveguide can be expressed as:

$$E(z,t) = E_1(z)e^{-i\omega_1 t + ik_1 z} + E_2(z)e^{-i\omega_2 t + ik_2 z} + E_3(z)e^{-i\omega_3 t + ik_3 z} + c.c. \quad (1)$$

We have ignored the polarization of the electrical fields for simplicity. Also assume the mode area for E_1, E_2 and E_3 are the same. The nonlinear polarization in the waveguide can be written as $P_{NL}(z,t) = \epsilon_0 \chi_2 E^2(z,t)$. The general nonlinear equation is:

$$\sum_j 2ik_j \frac{\partial E_j(z)}{\partial z} e^{-i\omega_j t + ik_j z} + c.c. = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2}. \quad (2)$$

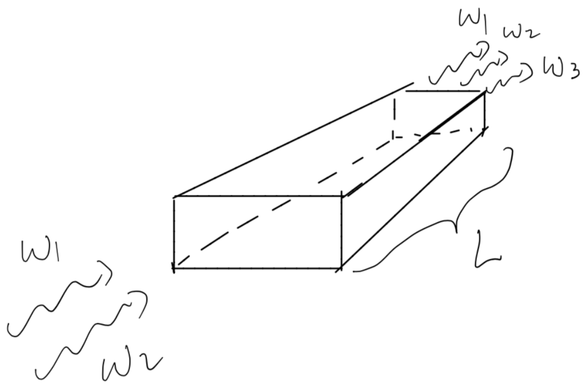

Figure 1: Waveguide for difference frequency generation.

(1) Write down the nonlinear equations of $E_1(z)$, $E_2(z)$ and $E_3(z)$ for difference frequency generation. (5 points)

(2) Show that your equations in question (1) are consistent with energy conservation. (10 points)

(3) Phase matching bandwidth for difference frequency generation:

Let's assume $n(\omega_1) = n(\omega_1/2) = 2$, and $|E_1(z)| \gg |E_2(z)|, |E_3(z)|$.And for frequency near $\omega_1/2$, we can expand the refractive index using Talyor expansion:

$$n(\frac{\omega_1}{2} + \Delta\omega) = n(\frac{\omega_1}{2}) + \left.\frac{dn}{d\omega}\right|_{\frac{\omega_1}{2}} \Delta\omega + \frac{1}{2}\left.\frac{d^2 n}{d\omega^2}\right|_{\frac{\omega_1}{2}} \Delta\omega^2. \quad (3)$$

Let's work in the regime where $|\omega_2 - \omega_3| \ll \omega_1/2$, $\left.\frac{dn}{d\omega}\right|_{\frac{\omega_1}{2}} = 10^{-13}$ s/rad and $\left.\frac{d^2n}{d\omega^2}\right|_{\frac{\omega_1}{2}} = 2 \times 10^{-29}\ s^2/rad^2$.

What is the maximum $|\omega_2 - \omega_3|$ that satisfies phase-matching condition? Here we define phase-matching condition as: $k_1 - k_2 - k_3 < 10^{-3} k_1$. (10 points)

(4) Consider the phase-matching bandwidth of second-harmonic generation using the same waveguide. Suppose the input frequency is $\omega_1/2 + \delta\omega$, the harmonic frequency is $\omega_1 + 2\delta\omega$. The refractive index near $\omega_1/2$ is the same as question (3) and $n(\omega_1/2) = n(\omega_1) = 2$. The refractive index around ω_1 can be expressed with another Taylor expansion:

$$n(\omega_1 + 2\delta\omega) = n(\omega_1) + \left.\frac{dn}{d\omega}\right|_{\omega_1} 2\delta\omega, \tag{4}$$

and $\left.\frac{dn}{d\omega}\right|_{\omega_1} = \left.\frac{dn}{d\omega}\right|_{\frac{\omega_1}{2}}$.

The phase-matching bandwith is referred to the maximum $|\delta\omega|$ that satisfies the phase matching condition for second-harmonic generation. Could you make a quick argument: is $|\delta\omega|_{max}$ larger or smaller than the $|\omega_2 - \omega_3|_{max}$ in question (3)? (5 points)

Question 2: Raman scattering (30 points)

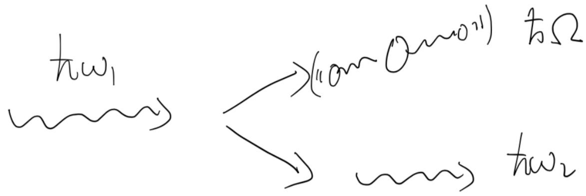

Figure 2: Raman process in this question.

In Raman scattering, the input photon ($\hbar\omega_1$) gives away certain amount of energy to the molecular vibration/rotation ($\hbar\Omega_1$), and outputs a photon with less energy ($\hbar\omega_2 = \hbar\omega_1 - \hbar\Omega$). In a waveguide,the equation of the incident field and the scattered field can be express as:

$$\frac{dE_1(z)}{dz} = -g_1 |E_2(z)|^2 E_1(z), \tag{5}$$

$$\frac{dE_2(z)}{dz} = g_2 |E_1(z)|^2 E_2(z), \tag{6}$$

, where $E(z,t) = E_1(z) e^{-i\omega_1 t + ik_1 z} + E_2(z) e^{-i\omega_2 t + ik_2 z} + c.c.$. In this question, you can define any parameters as you need, such as loss coefficient, cavity length, etc. For simplicity, assume the mode area and refraction index for E_1 and E_2 are the same.

(1) Given the fact that the photon number is conserved in Raman scattering, calculate $\frac{g_1}{g_2}$. (15 points)

(2) Raman laser. Let's put this waveguide with strong Raman gain inside a pair of mirrors (FP-cavity). Assume the cavity is on resonance with both ω_1 and ω_2.

Can you make a laser at frequency ω_2? What is the equation of motion for E_2? (equation of dE_2/dt) (10 points)

(3) What is the laser threshold? (5 points)

Question 3: Kerr parametric oscillator (40 points)

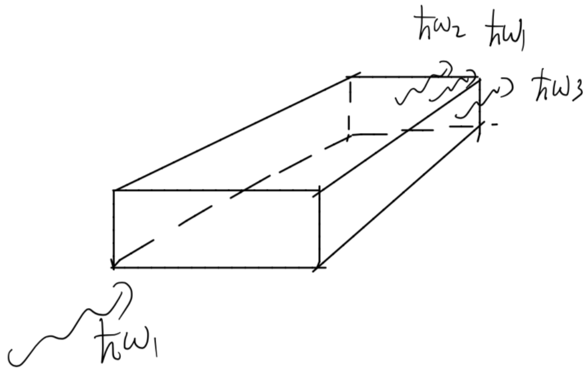

Figure 3: Waveguide for Kerr nonlinearity.

Consider Kerr effect inside a waveguide with length L. There is a very strong input E_1. Consider a three-wave situation, where

$$E(z,t) = E_1(z)e^{-i\omega_1 t + ik_1 z} + E_2(z)e^{-i\omega_2 t + ik_2 z} + E_3(z)e^{-i\omega_3 t + ik_3 z} + c.c.. \quad (7)$$

We have $2\omega_1 = \omega_2 + \omega_3$, and $|E_1| \gg |E_2|, |E_3|$. The nonlinear polarization in the waveguide can be written as $P_{NL}(z,t) = \epsilon_0 \chi_3 E^3(z,t)$. The boundary condition is that $E_1(z=0) = E_0$, $E_2(z=0) = E_3(z=0) = 0$.

(1) Write down the nonlinear equation for E_2 and E_3. Hint: use $|E_1| \gg |E_2|, |E_3|$. Only keep the leading nonlinear terms. (10 points).

(2) Put this waveguide between a pair of mirrors (FP-cavity). Assume the cavity is on resonance with ω_1, ω_2 and ω_3. To further simplify our calculation, we stay in the regime where $|\omega_2 - \omega_3| \ll \omega_1$. This allows us to let $\omega_1, \omega_2, \omega_3$ equal to ω_o and n_1, n_2, n_3 equal to n_o in our equations (Exception: keep the refractive index of n_1, n_2 and n_3 in the exponential phase matching terms. Otherwise you will have problem in section (3)).

Let $z = t \times c/n_o$, where c is the speed of light in vacuum. What is the equation of motion for $E_2(t)$ and $E_3(t)$ in the cavity? (10 points)

(3) In your equations in question (2), you should have a time dependent term that looks like this: $e^{i\delta\omega t}$. Can you remove this time dependent term in the equation? Hint: define $E_2(t) = E_2'(t)e^{ixt}$ and $E_3(t) = E_3'(t)e^{ixt}$. You should figure out what x is. (5 points)

(4) $|E_2(t)|, |E_3(t)|$ will have non-zero solution (oscillation, laser) when $|E_1|$ is above a certain threshold value. Solve this threshold value. (Note: at some point, you might have a quadratic equation for $|E_1|^2$. It is OK to stop there. The rest of the calculation is just messy math without much physics insight.) (10 points)

(5) Will you be able to have non-zero solution for $|E_2(t)|$ and $|E_3(t)|$ when you have perfect phase-matching condition? ($2k_1 = k_2 + k_3$, or equivalently $2n_1 = n_2 + n_3$ when we let $\omega_1, \omega_2, \omega_3$ equal to ω_o) (5 points)

Bonus Question: Vacuum Rabi oscillation (20 points)

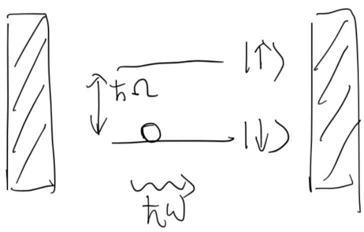

Figure 4: System in question 4.

Inside a FP cavity, we have an atom with two-level energy ($E_\uparrow - E_\downarrow = \hbar\Omega$). ω is the frequency of the photon in cavity. At time zero, the atom is at its ground state, and there is one photon in the cavity. Neglect all the loss in the system. You can define any parameters that you need.

(1) What is the Hamiltonian of the system? (5 points)

(2) If we express the wave function of our system at time t as:

$$\psi(t) = a(t)|\downarrow, 1\rangle + b(t)|\uparrow, 0\rangle, \tag{8}$$

where the 0 and 1 denotes the number of photon in the system. What is the equation of motion for a(t) and b(t)? (5 points)

(3) Assume $\Omega = \omega$. Solve a(t) and b(t). (10 points)