Week 6

Following previous class:

$$E\cos\omega t \quad \rightsquigarrow \qquad \overline{}\ |2\rangle \qquad E_2 - E_1 = \hbar\Omega$$
$$\overline{}\ |1\rangle$$

We found polarization : $P(t) = N\langle \psi(t)|\vec{\mu}|\psi(t)\rangle$

$$= N(\vec{\mu}_{12}\,\rho_{12}(t) + \vec{\mu}_{21}\,\rho_{21}(t))$$

$$\rho_{12}(t) = a^*(t)b(t)e^{-i\Omega t}$$

And we construct equations for $\rho_{12}(t)$ and includes spontaneous emission and phase decoherent.

$$\dot{\rho}_{12}(t) = -\left(i\Omega + \frac{1}{T_2}\right)\rho_{12}(t) + \dot{a}^*(t)b(t)e^{-i\Omega t} + a^*(t)\dot{b}(t)e^{-i\Omega t}$$

plug in

$$\dot{a}^*(t) = -\frac{i\vec{\mu}_{12}\cdot\vec{E}}{2\hbar}e^{i(\omega-\Omega)t}b^*(t)$$

$$\dot{b}(t) = \frac{i\vec{\mu}_{21}\cdot\vec{E}}{2\hbar}e^{-i(\omega-\Omega)t}a(t)$$

We have:

$$\partial_t \rho_{12} = \left(-i\Omega - \frac{1}{T_2}\right)\rho_{12}(t) - \frac{i\vec{\mu}_{12}\cdot\vec{E}}{2\hbar}(P_2(t) - P_1(t))e^{-i\omega t}$$

where $P_2(t) = |b|^2$: probability in excited state

$P_1(t) = |a|^2$: - - - - in ground state

And we can do integral to get the solution at $t \gg T_2$:

$$\rho_{12}(t) = \frac{-i\vec{\mu}_{12}}{2\hbar}\int_0^t dt'\, e^{-(i\Omega+\frac{1}{T_2})(t-t')}\vec{E}(t')e^{-i\omega t'}[\rho_2(t')-\rho_1(t')]$$

Because of $\frac{1}{T_2}$, the main contribution comes from $\frac{(t-t')}{T_2} \sim 1$

And since we assumed $\frac{t}{T_2} \gg 1$, so $\int_0^t \to \int_{-\infty}^t$

Next, apply week coupling condition:

$\Omega_R = \frac{\vec{\mu}\cdot\vec{E}}{\hbar} \ll \frac{1}{T_2}$: the speed of converting atom from $|1\rangle \to |2\rangle$ is much much slower than the decohent speed of the system

This is true for most classical system. (room temperature, thermal state)

What this means is that within the time scale T_2, $\vec{E}(t')$, $\rho_2(t')$, $\rho_1(t')$ does not change much.
(to change them, one need $\frac{1}{\Omega_R}$ time)

So we can simply use $\vec{E}(t)$ to replace $\vec{E}(t')$
because from $t-t' \sim t-T_2$ to t', $\vec{E}$ does not change much.

$\vec{E}(t') \to \vec{E}(t)$; $\rho_2(t') \to \rho_2(t)$; $\rho_1(t') \to \rho_1(t)$.

$$\rho_{12}(t) = -\frac{i\vec{\mu}_{12}\vec{E}(t)}{2\hbar}e^{-(i\Omega+\frac{1}{T_2})t}[\rho_2(t)-\rho_1(t)]\int_{-\infty}^t e^{i(\Omega-\omega)t'}\cdot e^{\frac{1}{T_2}t'}dt'$$

This is Rate equation Approximation.

calculate:

$$\int_{-\infty}^{t} e^{[i(\Omega-\omega)+\frac{1}{T_2}]t'} dt'$$

$$\int_{-\infty}^{a} e^{\alpha x} \cdot dx = \left.\frac{e^{\alpha x}}{\alpha}\right|_{-\infty}^{a} = \frac{1}{\alpha}\left\{e^{\alpha a} - e^{\alpha \cdot (-\infty)}\right\}$$

$$\therefore = \frac{1}{i(\Omega-\omega)+\frac{1}{T_2}}\left[e^{(i(\Omega-\omega)+\frac{1}{T_2})t} - e^{\{i(\Omega-\omega)+\frac{1}{T_2}\}\cdot(-\infty)}\right]$$

$\to 0.$

$$= \frac{e^{i(\Omega-\omega)+\frac{1}{T_2})t}}{i(\Omega-\omega)+\frac{1}{T_2}}$$

In combine: $$\rho_{12}(t) = -\frac{i\vec{\mu}_{12}\cdot\vec{E}}{2\hbar}\frac{[\rho_2(t)-\rho_1(t)]\, e^{(-i\Omega-\frac{1}{T_2})t}\, e^{[i(\Omega-\omega)+\frac{1}{T_2}]t}}{[i(\Omega-\omega)+\frac{1}{T_2}]}$$

$$= -\frac{i\vec{\mu}_{12}\cdot\vec{E}\,[\rho_2(t)-\rho_1(t)]\, e^{-i\omega t}}{2\hbar(i(\Omega-\omega)+1/T_2)}$$

$$\therefore \vec{P} = \vec{\mu}_{12}\cdot N\cdot \vec{\rho}_{12} + c.c.$$

$$= \frac{-i\vec{\mu}_{12}\cdot\vec{\mu}_{12}(N_2-N_1)}{2\hbar[i(\Omega-\omega)+\frac{1}{T_2}]}\cdot\vec{E}e^{-i\omega t} + c.c.$$

And $$\vec{P} = \frac{\varepsilon_0\chi(-\omega)\vec{E}e^{-i\omega t}}{2} + \frac{\varepsilon_0\chi(\omega)\vec{E}e^{i\omega t}}{2}$$

We have $$\boxed{\chi(-\omega) = \frac{-i}{\varepsilon_0\hbar}\left[\frac{N_2-N_1}{i(\Omega-\omega)+\frac{1}{T_2}}\right]\underbrace{\vec{\mu}_{12}\vec{\mu}_{21}^{T}}_{\text{Tensor.}}}$$

Later we will examine gain and loss, refractive index from this result.

A shortcut to get $\rho_{12}(t)$

$$\partial_t \rho_{12} = \left(-i\Omega - \frac{1}{T_2}\right)\rho_{12}(t) - \frac{i\vec{\mu}_{12}\cdot\vec{E}(t)}{2\hbar}\left(\rho_2(t) - \rho_1(t)\right)e^{-i\omega t}$$

Let $\rho_{12}(t) = \tilde{\rho}_{12}(t)e^{-i\omega t} + C_0$

Because of damping term, any initial condition $C_0 \to 0$ at $t \gg 0$.

$$\rho_{12}(t) = \tilde{\rho}_{12}(t)e^{-i\omega t}$$

$\boxed{\text{Rate equation approximation}}$

Assume $\vec{E}(t), \rho_2(t), \rho_1(t)$ varies much slower than $-\frac{1}{T_2}$ term.

So: $\frac{1}{T_2}\tilde{\rho}_{12}(t) \gg \dot{\tilde{\rho}}_{12}(t)$.

Then we can treat $\tilde{\rho}_{12}(t)$ as steady state. $\dot{\tilde{\rho}}_{12}(t) \to 0$.

$$\Rightarrow \quad -i\omega\,\tilde{\rho}_{12}(t)e^{-i\omega t} = \left(-i\Omega - \frac{1}{T_2}\right)\tilde{\rho}_{12}(t)e^{-i\omega t} - \frac{i\vec{\mu}_{12}\cdot\vec{E}(t)}{2\hbar}\left[\rho_2(t) - \rho_1(t)\right]e^{-i\omega t}$$

$$\Rightarrow \quad \tilde{\rho}_{12}(t)e^{-i\omega t} = \rho_{12}(t) = \frac{-i\vec{\mu}_{12}\cdot\vec{E}(t)\left[\rho_2(t) - \rho_1(t)\right]e^{-i\omega t}}{i(\Omega - \omega) + \frac{1}{T_2}}$$

Same result.

Refractive Index:

We know $\varepsilon_r = 1+\chi = n_r^2$

Assume χ is small, then:

$$n_r = (1+\chi)^{\frac{1}{2}} \approx 1+\frac{1}{2}\chi = 1 - \frac{i}{2\varepsilon_0\hbar}\left[\frac{N_2(t)-N_1(t)}{i(\Omega-\omega)+\frac{1}{T_2}}\right]\vec{\mu}_{12}\vec{\mu}_{21}^{\,T}$$

We are interested in the imaginary part: because it is connected with gain and loss.

$$\mathrm{Im}[n_r] = \frac{1}{2}\mathrm{Im}[\chi]$$

$$\therefore \chi = \frac{\vec{u}_{12}\cdot\vec{\mu}_{21}[N_2(t)-N_1(t)]}{\varepsilon_0\hbar} \frac{-i}{i(\Omega-\omega)+\frac{1}{T_2}}$$

$$= \cdots\cdots \frac{1}{(\omega-\Omega)+\frac{i}{T_2}}$$

$$= \cdots\cdots \frac{(\omega-\Omega)-\frac{i}{T_2}}{(\omega-\Omega)^2+1/T_2^2}$$

$$\therefore i\,\mathrm{Im}[n_r] = \frac{-i\vec{\mu}_{12}\vec{\mu}_{21}^{\,T}[N_2(t)-N_1(t)]/T_2}{2\varepsilon_0\hbar\,[(\omega-\Omega)^2+1/T_2^2]}$$

$\vec{u}_{12}\vec{\mu}_{21}^{\,T} \to |\vec{\mu}_{12}|^2$ (in our assumption)

So $N_2(t) - N_1(t) > 0$, $\quad Im[n_r] < 0$

$\quad \ldots\ldots < 0$, $\quad Im[n_r] > 0$.

In travelling wave:

$$E(z,t) = E e^{-i\omega t + ikz} = E e^{-i\omega t + i\frac{2\pi n_{real}}{\lambda}z} \cdot e^{-\frac{2\pi Im[n]}{\lambda}z}.$$

$$|E(z,t)|^2 = |E(z=0)|^2 e^{-\frac{4\pi Im(n)}{\lambda}z}$$

$$= |E(z=0)|^2 Exp\left\{ \frac{2\pi \mu_{12}^2 (N_2 - N_1)/T_2}{\varepsilon_0 \hbar [(\omega-\Omega)^2 + 1/T_2^2]} z \right\}.$$

When $N_2 - N_1 > 0$, Gain

$\quad \ldots\ldots < 0$, Loss.

If the gain is small, then we can expand Taylor series

$$|E(z,t)|^2 \approx |E(z=0)|^2 \left(1 + \frac{2\pi \mu_{12}^2 (N_2 - N_1)/T_2}{\varepsilon_0 \hbar [(\omega-\Omega)^2 + 1/T_2^2]} \cdot z \right)$$

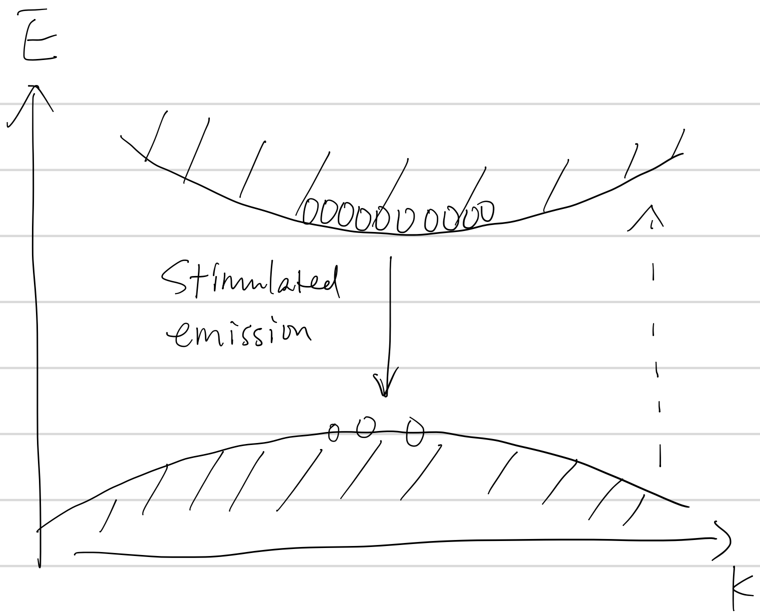
Linear increase with travelling distance.

We can define $g(N_2 - N_1) = g(\Delta N)$ is the gain.
and

$$|E(z,t)|^2 = |E(z=0)|^2 e^{g(\Delta N(t)) \cdot z}$$

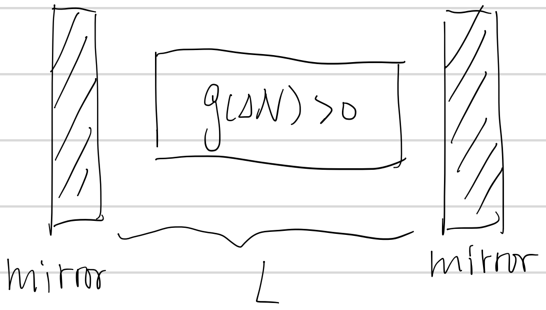
To make optical amplifier, or laser, we need to maintain $\Delta N(t) > 0$.

Example



ΔV: bias voltage to move electron to excited state

Laser: We are one step away from laser equation.



Let z be the distance of photon travel (not physical location).

For example, if a photon is reflected 10 times in the cavity, then $z = 10L$.

Let's define photon flux: $S = \frac{1}{2}\epsilon_0 |E|^2 / \hbar\omega$.

Why use photon instead of $|E|^2$?

Because when one photon is emitted, ΔN must reduce 1

$$\frac{dS}{dz} = \frac{d|E|^2}{dz} = g(\Delta N)\, S - \frac{1}{\Delta L} S$$

$\downarrow$ Loss of the cavity.
e.g. light lead through mirror.

Let $z = t \cdot c$ convert to time domain:

$$\frac{dS}{c\,dt} = g(\Delta N)\, S - \frac{S}{\Delta L}$$

$$\Rightarrow \frac{dS}{dt} = c \cdot g(\Delta N)\, S - \frac{c}{\Delta L} S$$

$$\frac{dS}{dt} = c \cdot g(\Delta N)\, S - \frac{1}{\tau} S.$$

Easy to see: $c \cdot g(\Delta N) - \frac{1}{\tau} > 0$; then $\frac{dS}{dt} > 0$.

photon number will keep growing: Laser.

Then when S stop growing?

Remember whenever a photon is emitted, ΔN has to reduce 1. Then ΔN reduce, g reduce, and finally reach steady-state $c \cdot g(\Delta N) - \frac{1}{\tau} = 0$. $\frac{dS}{dt} = 0$.

Now we need to construct equation of motion for $\Delta N(t)$

Because of pumping, we can define $\overline{\Delta N}$ as the population inversion when there's no cavity, no laser.
And when ΔN deviate from $\overline{\Delta N}$,
ΔN will slowly decay to $\overline{\Delta N}$.

A good example is spontaneous emission of an atom, where $\overline{\Delta N} = 0$
And : $\frac{d}{dt} \Delta N = -\frac{1}{T_1} \Delta N$. ; Solution is exponential decay.

Similarly, we will have:

$$\frac{d}{dt} (\Delta N) = -\frac{1}{T_1} (\Delta N - \overline{\Delta N}) + \dots$$

Another effect to consider is laser. When one photon is emit, ΔN must change by 2 (N_2 decreases by 1, N_1 increases by 1)

So: $\frac{d(\Delta N)}{dt} = -2c \cdot g(\Delta N) \cdot S - \frac{1}{T_1} (\Delta N - \overline{\Delta N})$

And from previous class, we have

$$g(\Delta N) = g' \cdot \Delta N.$$

Laser equation :

$$\frac{dS}{dt} = c \cdot g' \cdot \Delta N(t) \cdot S - \frac{1}{\tau} S$$

$$\frac{d\Delta N(t)}{dt} = -2cg'\Delta N(t) \cdot S - \frac{\Delta N(t) - \Delta \bar{N}}{T_1}$$

What's the steady state solution ?

$$c \cdot g' \cdot \Delta N(t) - \frac{1}{\tau} = 0. \quad \Rightarrow \quad \Delta N(t) = \frac{1}{cg'\tau}$$

plug into second equation:

$$-\frac{2}{\tau} S = \frac{\Delta N(t) - \Delta \bar{N}}{T_1} \quad \Rightarrow \quad S = \frac{\tau \left(\Delta \bar{N} - \frac{1}{cg'\tau}\right)}{2T_1}$$

$$S = \frac{\tau \Delta \bar{N} - 1/cg'}{2T_1}$$

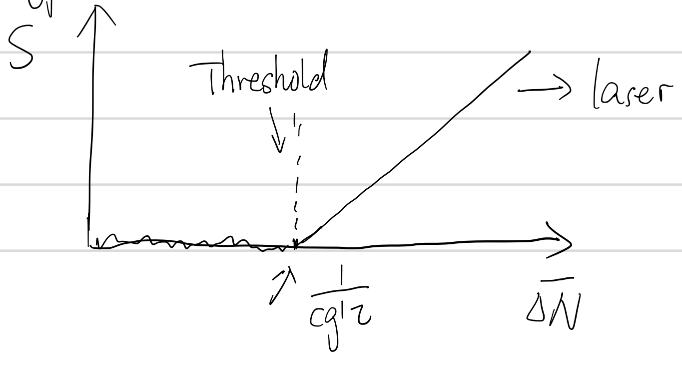
This is the number of photon in your laser.

$\Delta \bar{N} \uparrow$, $S \uparrow$: pump harder

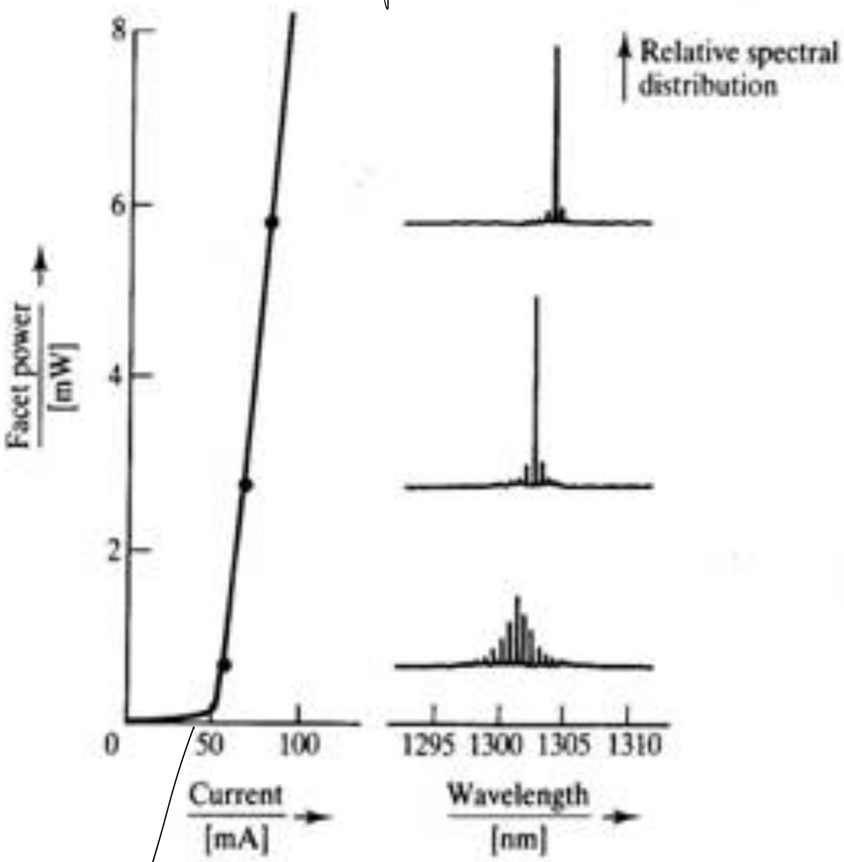
$\tau \uparrow$, $S \uparrow$: better cavity quality

$g' \uparrow$, $S \uparrow$: better gain material.

Typical laser evidence:



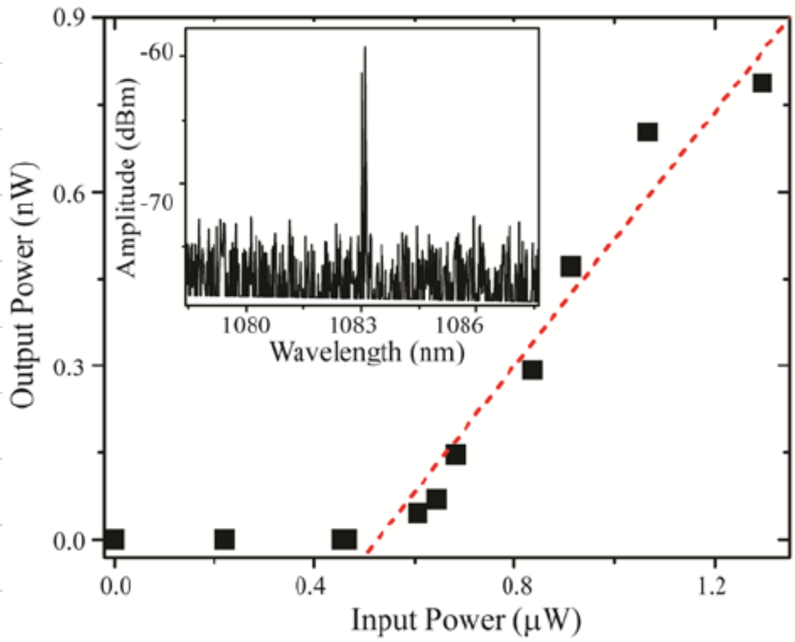
Electrical Pump



from Invocom.

Spontaneous emission

Optical pump



from Amani Group USC

Some final touch.

Is laser robust?

In another word, if we add a little noise to the steady state S_s, will S_s + Noise return to S_s or will it start to deviate from S_s?

This is called stability analysis.

$$\frac{dS}{dt} = cg'\Delta N\cdot S - \frac{1}{\tau}S$$

$$\frac{d\Delta N}{dt} = -2cg'\Delta N\cdot S - \frac{\Delta N - \Delta\bar{N}}{T_1}$$

Let $S = S_s + S'$; S_s is steady state photon number.

since S' is noise, $S' \ll S_s$.

Similarly: $\Delta N = \Delta N_s + \Delta N'$; $\Delta N' \ll \Delta N_s$

plug in:

$$\frac{dS_s}{dt} + \frac{dS'}{dt} = cg'(\Delta N_s + \Delta N')(S_s + S') - \frac{1}{\tau}(S_s + S')$$

$\frac{dS_s}{dt} = 0$ (steady state S_s is just a number)

$$\Rightarrow \frac{dS'}{dt} = cg'\Delta N_s\cdot S_s + cg'\Delta N_s\cdot S' + cg'\Delta N' S_s + cg'\Delta N'\cdot S' - \frac{S_s}{\tau} - \frac{S'}{\tau}$$

Notice for steady state S_s, we have $cg'\Delta N_s\cdot S_s = \frac{S_s}{\tau}$

In addition: $\Delta N'\cdot S'$ is much smaller than other term.

So: $$\frac{dS'}{dt} = cg'\Delta N_s \cdot S' + cg' S_s \cdot \Delta N' - \frac{S'}{\tau}$$

Use: $cg'\Delta N_s = \frac{1}{\tau}$

$$\Rightarrow \frac{dS'}{dt} = cg' S_s \cdot \Delta N'$$

Similarly, we have:

$$\frac{d\Delta N}{dt} = -2cg'\Delta N \cdot S - \frac{\Delta N - \Delta\bar{N}}{T_1}$$

$$\frac{d\Delta N'}{dt} = -2cg'\Delta N_s \cdot S_s - 2cg'\Delta N_s \cdot S' - 2cg'\Delta N' \cdot S_s - 2cg'\Delta N' S' - \frac{\Delta N_s - \Delta\bar{N}}{T_1} - \frac{\Delta N'}{T_1}$$

$$= -2cg'\Delta N_s \cdot S' - 2cg' S_s \cdot \Delta N' - \frac{\Delta N'}{T_1}$$

Use: $-2cg' S_s \cdot \Delta N_s = \frac{\Delta N_s - \Delta\bar{N}}{T_1} \Rightarrow -2cg' S_s = \frac{\Delta N_s - \Delta\bar{N}}{T_1 \cdot \Delta N_s} = \frac{1}{T_1} - \frac{\Delta\bar{N}}{\Delta N_s} \cdot \frac{1}{T_1}$

$$\Rightarrow \frac{d\Delta N'}{dt} = -2cg'\Delta N_s \cdot S' + \frac{1}{T_1}\Delta N' - \frac{\Delta\bar{N}}{\Delta N_s}\frac{1}{T_1} - \frac{\Delta N'}{T_1}$$

$$= -2cg'\Delta N_s \cdot S' - \frac{\Delta\bar{N}}{\Delta N_s}\frac{1}{T_1}$$

In sum: $$\frac{d\Delta N'}{dt} = -2cg'\Delta N_s \cdot S' - \frac{\Delta\bar{N}}{\Delta N_s} \cdot \frac{\Delta N'}{T_1}$$

$$\frac{dS'}{dt} = cg' S_s \cdot \Delta N'$$

Solve it:

$$\frac{d\Delta N'}{dt} = -2cg'\Delta N_s \cdot S' - \frac{\overline{\Delta N}}{\Delta N_s} \cdot \frac{\Delta N'}{T_1}$$

$$\frac{dS'}{dt} = cg' S_s \cdot \Delta N'$$

Take $\frac{d}{dt}$ on the first equation:

$$\partial_t^2 \Delta N' = -\frac{\overline{\Delta N}}{\Delta N_s T_1} \cdot \partial_t \Delta N' - 2cg'\Delta N_s \cdot \partial_t S'$$

plug second equation into the above equation:

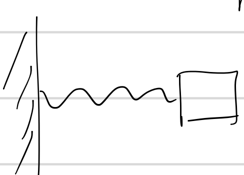
$$\partial_t^2 \Delta N' + \frac{\overline{\Delta N}}{\Delta N_s T_1} \cdot \partial_t \cdot \Delta N' + 2c^2 g'^2 \cdot S_s \cdot \Delta N_s \cdot \Delta N' = 0$$

It's the same form as

$$\partial_t^2 \Delta N' + b \cdot \partial_t \Delta N' + c \cdot \Delta N' = 0 . \quad ; \quad b, c > 0.$$

This is the same equation as a harmonic oscillator with drag:

$$m\frac{d^2x}{dt^2} = ma = -kx - \gamma \cdot \frac{dx}{dt}$$

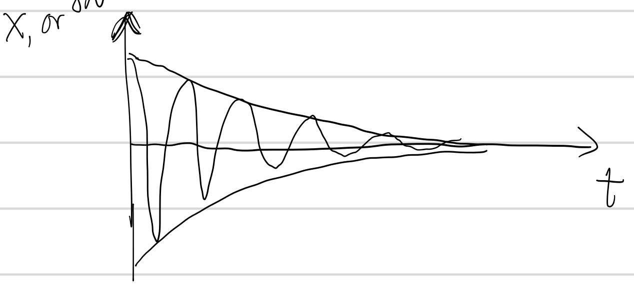


$$\Rightarrow \frac{d^2x}{dt^2} + \frac{\gamma}{m} \cdot \frac{dx}{dt} + kx = 0.$$

This system gives energy away, x will slow down and stop eventually because of the drag.

So: $x(t \to \infty) \to 0.$

So: $\Delta N'(t \to \infty) \to 0.$



We can also solve it. Let $\Delta N' = \Delta N_0' e^{+\alpha t}$.

$$\Rightarrow \alpha^2 \Delta N' + b\cdot\alpha\cdot\Delta N' + c\Delta N' = 0$$

$$\therefore \alpha^2 + b\alpha + c = 0; \quad b>0, \; c>0.$$

$$\alpha = \frac{-2b \pm \sqrt{b^2-4c}}{2};$$

If $b^2-4c<0$; $\alpha = -b \pm i\frac{\Omega}{2}$; $\Rightarrow \Delta N' = \Delta N_0' e^{i\Omega t/2}\cdot e^{-bt}$ or $\Delta N_0' e^{-i\Omega t/2} e^{-bt}$

If $b^2-4c>0$. Just decay.

$$b = \frac{\bar{\Delta N}}{\Delta N_s}\cdot\frac{1}{T_1}; \quad c = 2c^2 g'^2 S_s \Delta N_s$$

$$b^2-4c = \frac{\bar{\Delta N}^2}{\Delta N_s^2}\cdot\frac{1}{T_1^2} - 8c^2g'^2 S_s\Delta N_s = \frac{\bar{\Delta N}^2 - 8c^2g'^2 S_s\cdot\Delta N_s^3 T_1^2}{\Delta N_s^2 T_1^2}$$

Use: $cg'\Delta N_s\cdot S_s = \frac{1}{2}S_s$; $b^2-4c \propto \bar{\Delta N}^2 - 8cg' S_s\cdot \Delta N_s^2 T_1^2/2$

Use again: $cg'\Delta N_s\cdot S_s = \frac{1}{2} S_s$; $b^2-4c\propto \bar{\Delta N}^2 - 8\Delta N_s\cdot S_s\cdot T_1^2/2^2$

Now use $2cg'\Delta N_s\cdot S_s = \frac{\bar{\Delta N}-\Delta N_s}{T_1} = \frac{2S_s}{2}$ $\therefore$ $2S_s = \frac{(\bar{\Delta N}-\Delta N_s)2}{T_1}$

$$\Rightarrow b^2-4c \propto \bar{\Delta N}^2 - 4\Delta N_s(\bar{\Delta N}-\Delta N_s)\frac{T_1}{2} = \bar{\Delta N}^2 + 4\frac{T_1}{2}\cdot\Delta N_s^2 - 4\frac{T_1}{2}\bar{\Delta N}\cdot\Delta N_s$$

$$b^2 - 4c \propto \bar{\delta N}^2 + \frac{4T_1}{\tau}\delta N_s^2 - \frac{4T_1}{\tau}\bar{\delta N}\delta N_s$$

For $T_1 < \tau$ (fast gain relaxation, slow cavity photon decay) ; $\frac{T_1}{\tau} < 1$;

$$\Rightarrow \left(\frac{T_1}{\tau}\right)^2 < \frac{T_1}{\tau} \Rightarrow \frac{T_1}{\tau} < \sqrt{\frac{T_1}{\tau}}$$

$$b^2 - 4c \propto \bar{\delta N}^2 + \frac{4T_1}{\tau}\delta N_s^2 - \frac{4T_1}{\tau}\bar{\delta N}\delta N_s > \bar{\delta N}^2 + \frac{4T_1}{\tau}\delta N_s^2 - 4\sqrt{\frac{T_1}{\tau}}\,\bar{\delta N}\delta N_s$$

$$= \left(\bar{\delta N} - 2\sqrt{\frac{T_1}{\tau}}\,\bar{\delta N}\delta N_s\right)^2 > 0.$$

$$\therefore \quad \delta N'(t) = \delta N_0\, e^{\alpha t}$$

$$\alpha = -\frac{\bar{\delta N}}{\delta N_s}\cdot\frac{1}{T_1} \pm \frac{\sqrt{b^2 - 4ac}}{2}$$ is real, $\alpha < 0$ decay.

So: $\delta N'(t) = \delta N'(t=0)\, e^{-\beta t}$; $\beta > 0$;

Direct decay. No oscillation at all.

$$c g' \delta N_s \cdot S' = -\frac{d\delta N'}{dt} - \frac{\bar{\delta N}}{\delta N_s}\frac{\delta N'}{T_1}$$

$$\Rightarrow S'(t) = -\beta\,\delta N' - \frac{\bar{\delta N}}{\delta N_s}\frac{1}{T_1}\delta N' = -\left(\beta + \frac{\bar{\delta N}}{\delta N_s T_1}\right)\delta N_0\, e^{-\beta t} ; \beta > 0.$$

So the laser is robust to perturbation.