Week 1

Quantum Mechanics

Target: know for sure that quantum mechanics is absolutely different from anything in the classical physics, and might be useful for computing and information in the future.

It's a hard task. There's no clear message

Unlike other scientific breakthrough in 20th century:
General relativity
genes, DNA.

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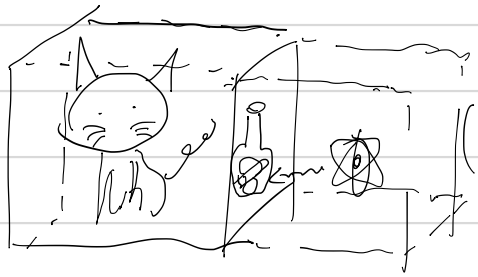
what about
Quantum mechanics?
Uncertainty principle?

Schordinger's Cat? ✡ pick up line...



$t \sim 1$ hour

(50%, 50%) Coin

Create two paradox:

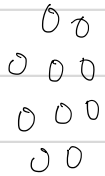
1. Our observation determines cat's live/death.
   Paradox: Then whom determine ours?
2. The cat is simultaneous live/dead.

So how do we end up here? Why Sch... do this to his cat?

It's estabished to express Sch... frustration on the quantum theory he co-founded. The quantum theory was based on series of experiments.

Two properties

Electron — Color: B / W — Orientation: Head / Tail



Electron band

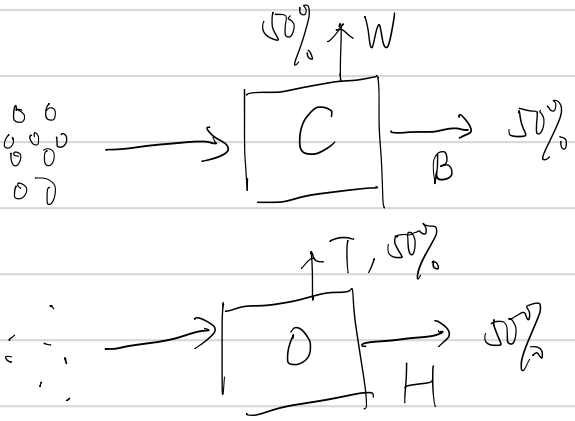
Electron bank:

Randomness: 50% B, 50% W

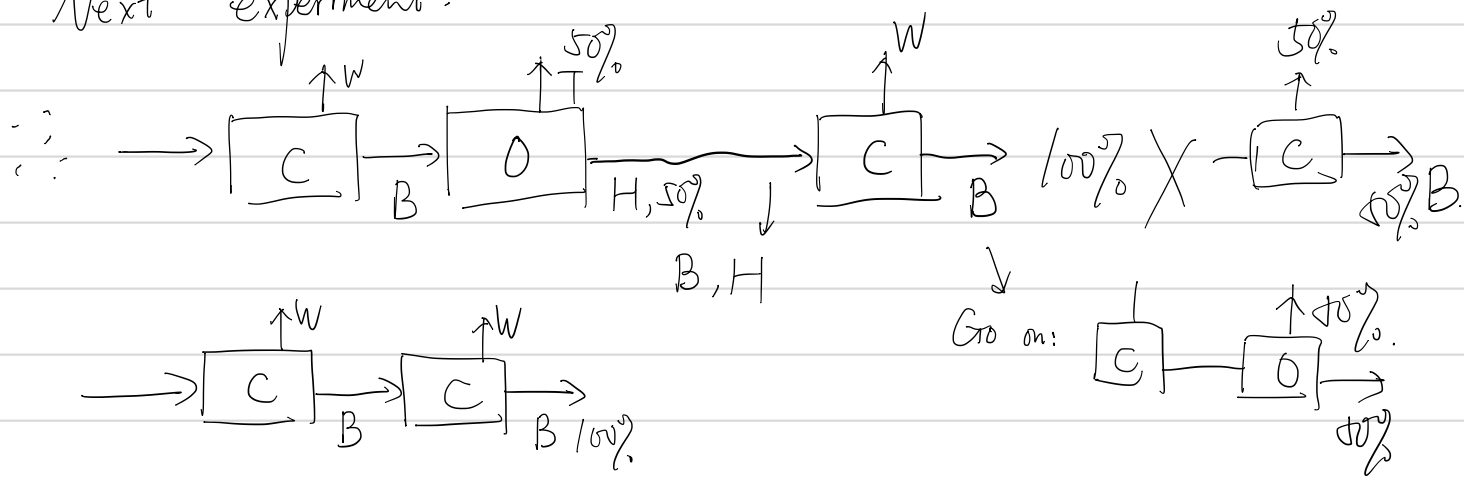
50% H, 50% T 50% H 50% T.

Classically, this is a simple case:

25%, B, H; 25% B, T; 25% W, H; 25% W, T.



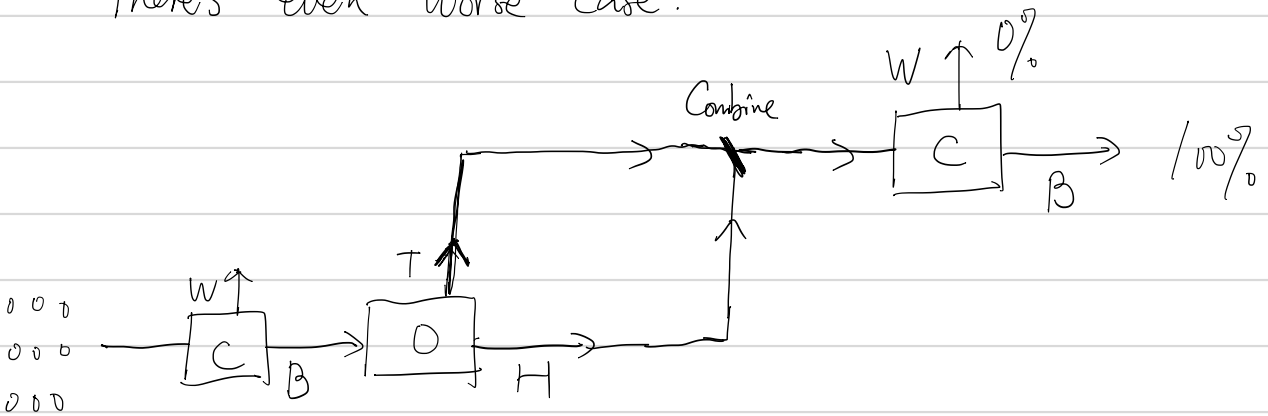
Next experiment:



Does the orientation box change the color?
But wait, why it changes the color in an probability way?

That's terrible...

There's even worse case.



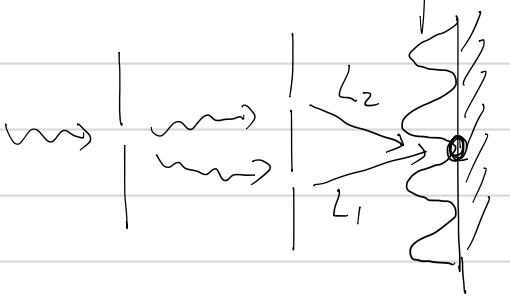
If one path is blocked, it's 50%, 50%.
Now combine two 50% 50%, what you would get?

So you might wonder, this could comes from electron-electron interactions.

Now how about send a single electron

If I block one path, what would happen?
50%, 50%.

How is this possible? Remined ourselves of



block one path: it's bright.
another path: also bright.
when open both path: dark.

$$|E_x|^2 = |E_0 e^{ikL_1} + E_0 e^{ikL_2}|^2$$ // Light goes through both path.

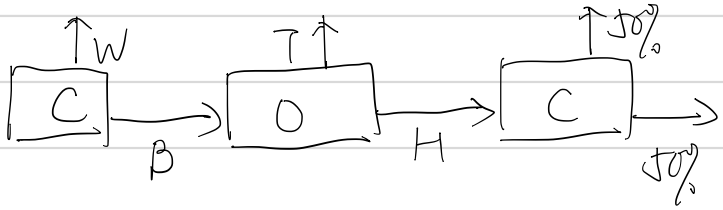
The terrifying experiments stop here. So is single electron!

Then what's the take away here:

1°. Classical math/physics just don't work.

2°. Two principles were derived from here:

A.



Uncertainty principles: You cannot know color and orientation at the same time.

$$\Delta O \cdot \Delta C \geq \delta.$$

B. Superpositim: Interference experiment. (electron goes through both path!)

$$|T\rangle = |B\rangle + |W\rangle \qquad |T\rangle + |H\rangle = 2|B\rangle.$$
$$|H\rangle = |B\rangle - |W\rangle$$

Indeed, the electrons are both black and white, Just like the damn Schordinger's cat.

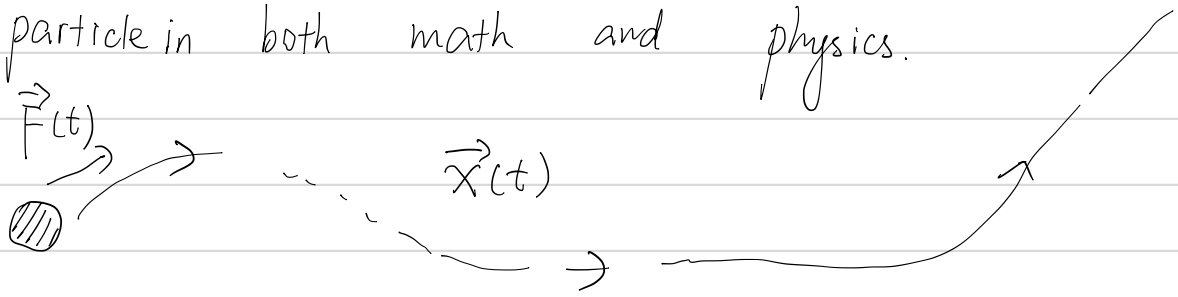
Fundamental of Waves.

Examples of waves ?

Sound, Light, water wave, mechanical wave, ...

Something in common ? They propagate, at a certain speed, they carry energy, ...

Why we need to study waves ? They are fundamentally different from particle in both math and physics.



How to describe a particle ? It has mass, where it is (position) and how is it moving (velocity).

We can describe the dynamics of a particle by recording its position at any given time:

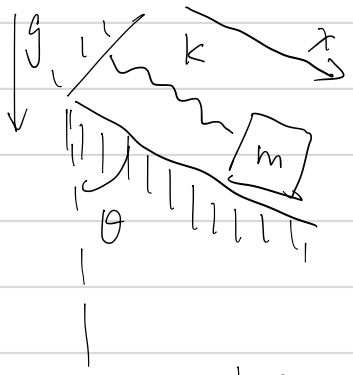
$$\vec{x}(t) = (x(t), y(t), z(t))$$

And what determines this motion?

$$\vec{F}(t) = m\vec{a} = m \cdot \frac{d\vec{v}(t)}{dt} = m \cdot \frac{d^2\vec{x}(t)}{dt^2}$$

If you know $\vec{F}(t)$, the force; $\vec{x}(t=0)$, initial position; $\frac{d\vec{x}}{dt}(t=0)$, initial speed, you can solve its motion $\vec{x}(t)$.

Interestingly, if your system conserves energy, meaning all forces are coming from potential energy (like gravity, a spring, ... etc.); simply knowing the total energy expression of the system can determine $\vec{x}(t)$.



$$E = -m \cdot g \cdot (x\cos\theta) + \frac{1}{2}k(x-x_0)^2 + \frac{1}{2}mv^2.$$

$$= E_p + E_k$$

E_p: potential energy; E_k: kenetic energy

Why? Because you can convert potential energy into force, and kenetic energy into inertial.

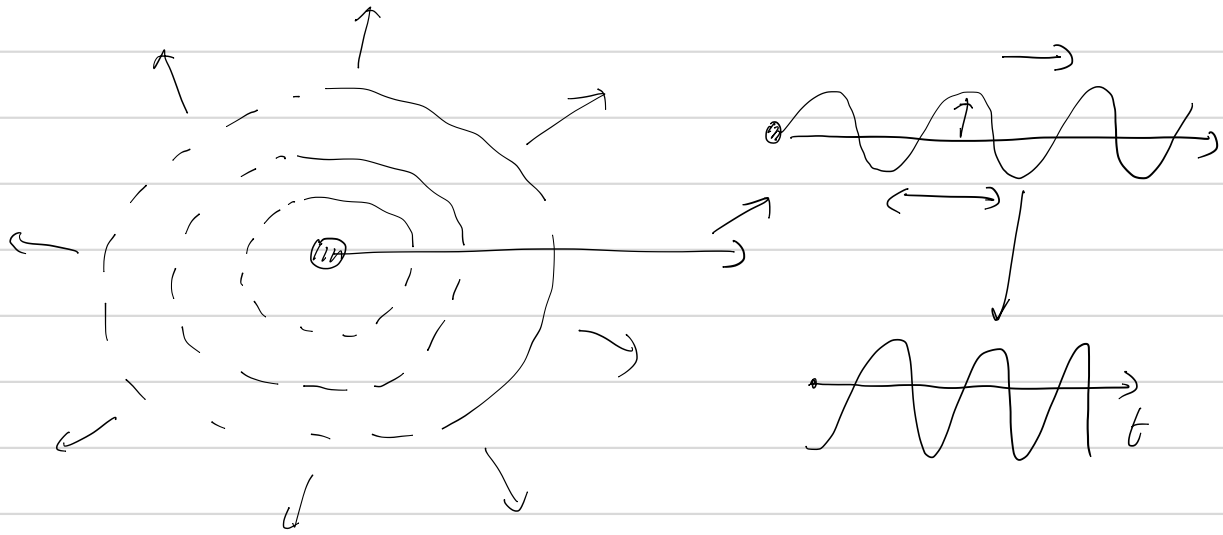
$$F = -\frac{\partial E}{\partial x} = mg\cos\theta - k(x-x_0)$$

$$ma = \frac{\partial}{\partial t}\left(\frac{\partial E}{\partial v}\right) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial v}\left(\frac{1}{2}mv^2\right)\right) = \frac{\partial}{\partial t}(mv) = ma$$

$$\frac{\partial E}{\partial x} + \frac{\partial}{\partial t}\left(\frac{\partial E}{\partial v}\right) = 0. \iff F = ma$$

This type of method, in general, is called analytical mechanics. It's very useful for complicated systems.

Now how to describe a wave?



Amplitude, wave speed, wavelength, frequency.

Are they constant for a certain wave? Maybe., at least amplitude is not constant here.

So how to describe a wave mathematically? $\vec{x}(t)$?

Amplitude at any position at any time.

$$A(\vec{x}, t) = A(x, y, t)$$

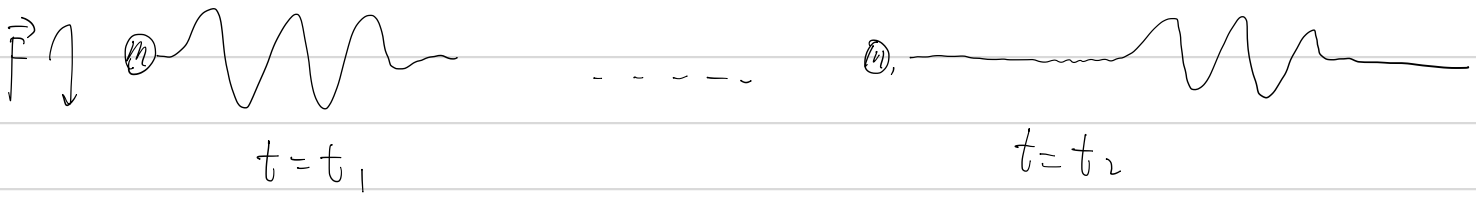
We can derive wave speed, amplitude, frequency from this expression.

particle		wave
$\vec{x}(t) = (x(t), y(t), z(t))$	$\Leftrightarrow$	$A(\vec{x}, t) = A(x, y, z, t)$
$\vec{F} = m\vec{a} = m \cdot \frac{d^2\vec{x}}{dt^2}$	$\Leftrightarrow$	equation for $A(\vec{x}, t)$?

$$\frac{\partial E}{\partial x} + \frac{\partial}{\partial t}\left(\frac{\partial E}{\partial v}\right) = 0. \Leftrightarrow$$ wave equation 2 ?

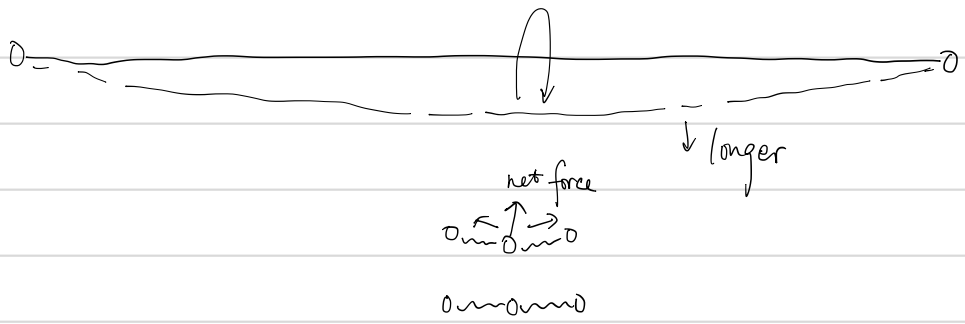
To answer the question of what is wave equation, we must notice one thing:

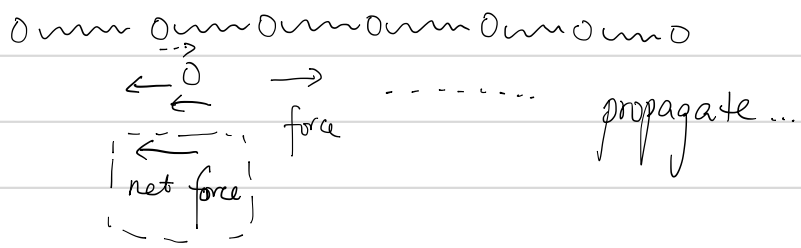
wave continue to propagate and its amplitude continue to change after you stop perturbing the water



So wave propagation (wave equation) can be independent of $\vec{F}$

What force causes the water far away from the source to move?





From this very simple picture of how wave propagates, we can make some amazing predictions of speed of sound.

For faster speed of sound, we would want the atoms/molecules to move easily. So <u>low mass</u>, and high stress <u>k</u>. $\left(F = \underset{\text{low}}{m}a = \underset{\text{high.}}{k} \cdot \Delta x \right)$

k comes from : interaction strength between molecules/atoms.

air, < water, < ice

340 m/s 1500 m/s 4000 m/s

What natural material has the highest speed of sound?

low mass high bonding strength (hard)

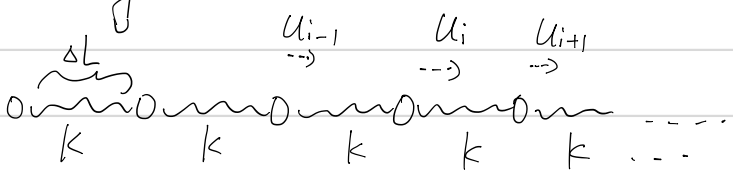
periodic table of element:

H, He, Li, B, C, N, O, F, Ne ...

△

Diomand. (12,000 m/s).

Wave equation: ⟶ x. $u(x,t)$



Amplitude in this wave is its displacement. $u_1, u_2, \dots u_i, \dots$

$$F_i = -k(u_i - u_{i+1}) - k(u_i - u_{i-1}) = ma_i = m\cdot\frac{d^2u_i}{dt^2}$$

$$= k\{(u_{i+1} - u_i) - (u_i - u_{i-1})\}$$

$$= k\{\underbrace{(u_{i+1} - u_i)}_{\Delta u_i} - \underbrace{(u_i - u_{i-1})}_{\Delta u_{i-1}}\}$$

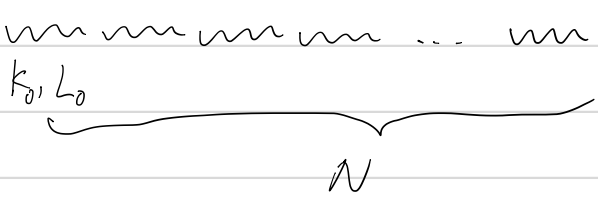
$$= k\underbrace{\{\Delta u_i - \Delta u_{i-1}\}}_{\Delta(\Delta u)}$$

$$u_{i+1} - u_i \approx \left.\frac{\partial u}{\partial x}\right|_{x=x_i}\cdot\Delta L \quad ; \quad (u_i - u_{i-1}) \approx \left.\frac{\partial u}{\partial x}\right|_{x=x_{i-1}}\cdot\Delta L$$

And: $\left(\left.\frac{\partial u}{\partial x}\right|_{x=x_i} - \left.\frac{\partial u}{\partial x}\right|_{x=x_{i-1}} \right) \triangleq \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=x_i} \cdot \Delta L$

$\therefore F_i = k \cdot \Delta L^2 \cdot \left.\frac{\partial^2 u}{\partial x^2}\right|_{x=x_i} = m \cdot \left.\frac{\partial^2 u}{\partial t^2}\right|_{\text{at } x=x_i}$

$\underbrace{k \cdot \Delta L}_{\text{Hooke's strain Constant}} \cdot \frac{\partial^2 u}{\partial x^2} = \underbrace{\frac{m}{\Delta L}}_{\text{one-dimensim density}} \cdot \frac{\partial^2 u}{\partial t^2}$; $u = u(x,t)$



Total $k = \frac{k_0}{N}$

$\therefore k \cdot N \cdot L_0 = k_0 \cdot L_0$

$\Rightarrow k \cdot L = \text{Constant}$.

Young's Modulus

$\Rightarrow \underbrace{\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho} \cdot \frac{\partial^2 u}{\partial x^2}}$

A typical wave-equation.

differential on both time, and positim.

Differential on time $\longrightarrow$ ma

Differential on space $\longrightarrow$ force.

Solution to wave equation ?

(Need to remember) ; There's a series of them.

$u_w(x,t) = u_0 \cos(\omega t - kx)$; or $\sin(\omega t - kx)$, $\cos(\omega t + kx)$, $\cos(\omega t - kx)$

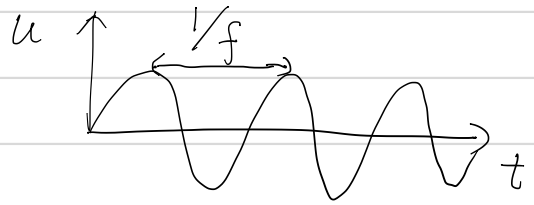
as long as:

$$-\omega^2 = -\frac{E}{\rho} \cdot k^2 \Rightarrow k = \sqrt{\frac{\rho}{E}} \cdot \omega.$$

1°.

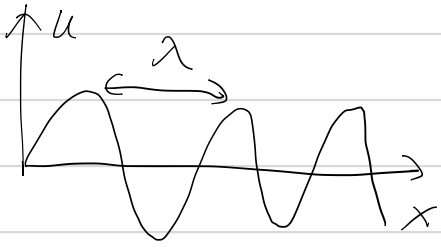
What is ω? Angular frequency, or at a certain position, how much 1/time you need for phase to change from $0 \to 2\pi$.

$\frac{\omega}{2\pi} = f$: frequency;



What is k? Wave-vector:

$k = \frac{2\pi}{\lambda}$; λ is wavelength



for a wave at a fixed time

$\frac{2\pi}{\lambda} \cdot \lambda = 2\pi \to$ phase $0 \to 2\pi$.

2° We now have time, have distance, seems that we should be able to get speed ?

At time $t = \frac{1}{f} = \frac{2\pi}{\omega}$; wave moves forward one wavelength λ. (by looking at the peak location, as $t \to t + 1/f$, amplitude is unchanged, then wave has to move forward by λ.

$$v = \frac{\lambda}{1/f} = \frac{\lambda\omega}{2\pi} = \frac{\omega}{k}$$

What is $\frac{\omega}{k}$? $\frac{\omega}{k} = \sqrt{\frac{E}{\rho}} = v$.

So: wave-equation: $\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$

3°. Interestingly, any given ω is a solution to the equation.

$$u_1 = |u_1| \sin(\omega_1 t - k_1 z) \; ; \; u_2 = |u_2| \sin(\omega_2 t - k_2 z) + \ldots\ldots$$

And $|u_1|, |u_2|, \ldots |u_N|$ can be arbitary number

Even more interestingly, $u = u_1 + u_2$ is still a solution to the wave-equation.

$$\frac{\partial^2 u}{\partial t^2} - v^2 \frac{\partial^2 u}{\partial x^2} = 0$$

So: $\frac{\partial^2}{\partial t}(u_1 + u_2 + \ldots u_N) - v^2 \frac{\partial^2 (u_1 + u_2 + \ldots u_N)}{\partial x^2} =$

$$\left(\frac{\partial^2 u_1}{\partial t^2} - v^2\frac{\partial^2 u_1}{\partial x^2}\right) + \left(\frac{\partial^2 u_2}{\partial t^2} - v^2\frac{\partial^2 u_2}{\partial x^2}\right) + \cdots \left(\frac{\partial^2 u_N}{\partial t^2} - v^2\frac{\partial^2 u_N}{\partial x^2}\right)$$

$$= \underbrace{0 + 0 + 0 + \cdots + 0}_{N} = 0.$$

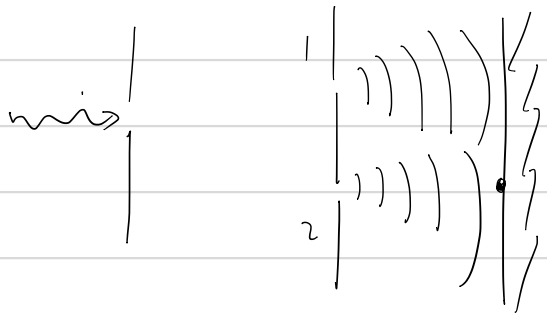
Always satisfied.

For linear wave equation, any linear combination is a solution.

$$u(x,t) = \sum_{i=1}^{N}\{a_1 \sin(\omega_i t - k_i z) + a_2 \sin(\omega_i t + k_i z) + a_3 \cos(\omega_i t - k_i z) + a_4 \cos(\omega_i t + k_i z)$$

$$= \int_{-\infty}^{+\infty}\{a_1(\omega)\sin(\omega t - k_\omega z) + a_2(\omega) \cdots\} d\omega$$

That's why we can add the wave together at Young's Double slit experiment



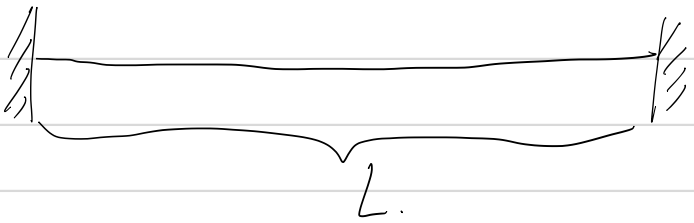
When is wave linear? As long as there's no terms that introduce interaction

like u^2, u^3, $\frac{\partial}{\partial t}u^2$, $\left(\frac{\partial}{\partial t}u\right)\left(\frac{\partial}{\partial x}\cdot u\right)$

4°. If any frequency is a solution,

Then how come guitar/violin has a certain sound?

We need "boundary condition"



Boundary condition: $u(x=0) = u(x=L) = 0$

At any given time.

At $x=0$, you need $u \to 0$, so:

$u_w = \sin(\omega t - kx) - \sin(\omega t + kx)$; or $\cos(\omega t - kx) - \cos(\omega t + kx)$

Any of this pair, satisfy $u(x=0) = 0$.

Use $\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\sin\left(\frac{\alpha-\beta}{2}\right)$; $\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\sin\left(\frac{\alpha-\beta}{2}\right)$

$\therefore$ $u_{1,\omega} = \sin(\omega t - kx) - \sin(\omega t + kx) \to \cos\omega t \cdot \sin kx$

$u_{2,\omega} = \cos(\omega t - kx) - \cos(\omega t + kx) \to \sin\omega t \cdot \sin kx$.

Then u_N also needs to satisfy $u(x=L) = 0$.

So: $kL = N \cdot \pi$ $\quad k_N = \frac{\pi N}{L} = \frac{\omega_N}{v} = \frac{2\pi f_N}{v}$

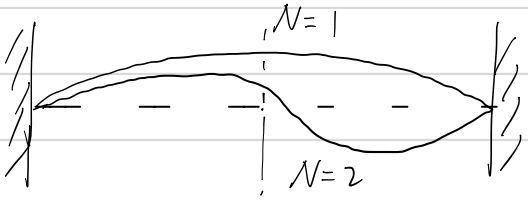
$\Rightarrow \quad f_N = \frac{v \cdot N}{2L}$: eigen-frequency.

Tune the string tighter, $E \uparrow$, $v = \sqrt{\frac{E}{\rho}} \uparrow$; $f_N \uparrow$.

This is how one tunes the frequency.

$$u(x,t) = \sum_j \sin(j \cdot k_0 x) \{ a_j \sin(j\omega_0 t) + b_j \cos(j\omega_0 t) \}$$

$$k_0 = \frac{\pi}{L} \qquad \omega_0 = \frac{\pi v}{L}$$



5°. The question continues: from the solution above, then our guitar should have a mixed sound of ω_0, $2\omega_0$, $3\omega_0$, ...

ω_0, $2\omega_0$, $3\omega_0$, ... same pitch, by octave. (perfect intervals)

Answer? Initial condition.

When you play your instrument, at the time you let go, you creat a waveform:

$u(x, t=0) = y_0(x)$; A function.

Also it has some speed.

$\frac{\partial u}{\partial t}(x, t=0) = z_0(x)$; Another function.

So the solution:

$$u(x,t) = \sum_j \sin(j \cdot k_0 x) \{ a_j \sin(j\omega_0 t) + b_j \cos(j\omega_0 t) \}$$

Different initial condition $\longrightarrow$ different a_j and b_j

possible to make $a_1, b_1 \gg a_2, b_2, a_3, b_3, \ldots$

Solve u_j such that $u(x,t)\big|_{t=0} \to y_0(x)$

$$\frac{\partial u}{\partial t}(x, t=0)\Big|_{t=0} \to z_0(x).$$

At $t=0$, as $\sin(j\omega_0 t) = 0$; So: $u(x,t)\big|_{t=0} \to y_0(x)$ solves b_j

At $t=0$, as $\frac{\partial}{\partial t}\cos(j\omega_0 t) = 0$; $\frac{\partial u}{\partial t}(x, t=0) \to z_0(x)$ solves a_j

Summary of wave:

1°. Description: amplitude of all locations and time.

$A(x, y, z, t)$

2°. wave equation? differential of t and x.

General way:

$$\left(a_1 \frac{\partial}{\partial t} + a_2 \frac{\partial^2}{\partial t^2} + b_1 \frac{\partial}{\partial x} + b_2 \frac{\partial^2}{\partial x^2}\right) A(x,t) = 0.$$

One dimension.

3°. For linear waves: (No interaction among waves)
The linear combition of waves are still solutions.

4° Need boundary condition to solve eigenstate.

5°. Need initial conditin to solve full wave-solution.