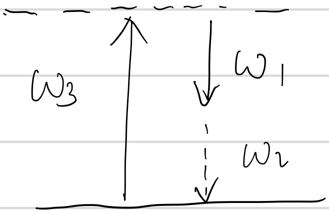
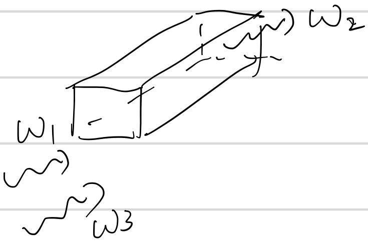
Difference frequency generation, parametric amplifier, and optical parametric oscillator.





Input ω_1 and ω_3, get difference frequency at $\omega_2 = \omega_3 - \omega_1$

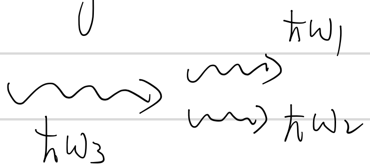
First prediction: if E_3 is much stronger than E_1, then compare the input port and output port, Does E_1 increase along z?

$|E_1(z=L)|$ $<, >, =$? $|E_1(z=0)|$?

Assume phase-matched, then $|E_1(z)|$ increases with z.

Why?

Everytime a $\hbar\omega_2$ photon is generated, a $\hbar\omega_1$ photon is also generated.



So the behavior of $E_1(z)$ is very different when compared with sum-frequency.

E_1 is not consumed, but rather triggers another generation of $\hbar\omega_1$ photon

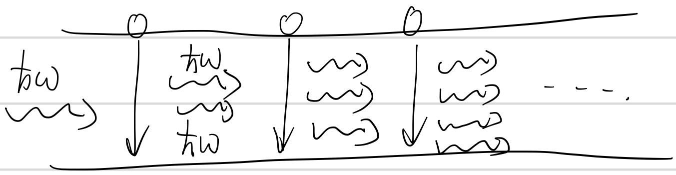
And what does that affect $E_2(z)$?

The growth rate of $E_2(z)$ shall increase with $E_1(z)$, and therefore z.

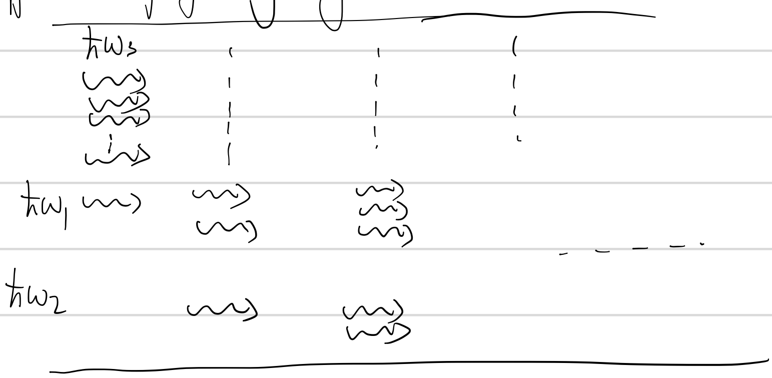
$$\frac{\partial |E_2(z)|}{\partial z} > 0.$$

So this is very similar to an amplifier.

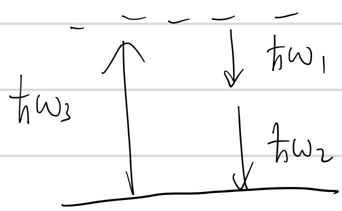
Fiber amplifier



Difference frequency generation



So Difference frequency generation is also called parametric amplifier. Why "parametric"? The state of the material is unchanged.



Now let's formulate this process.

$$\sum_{j=1,2,3} \vec{e_j}\, 2ik_j \frac{\partial E_j(z)}{\partial z} e^{-i\omega_j t + ik_j z} + c.c. = \mu_0 \frac{\partial^2 \vec{P}_{NL}}{\partial t^2}$$

$\vec{P}_{NL} = \varepsilon_0 \overleftrightarrow{\chi}^{(2)} : \vec{E}\vec{E}$; Again, we simplify this by assuming all the fields have the same polarization: $\vec{e_j} = \vec{e_0}$.

$$\vec{P}_{NL} = \vec{e_0}\,\varepsilon_0\,\chi^{(2)}\left(E_1 + E_2 + E_3 + E_1^* + E_2^* + E_3^*\right)^2$$

Only collect the terms with $\omega_1 = \omega_3 - \omega_2$; $\omega_2 = \omega_3 - \omega_1$; $\omega_3 = \omega_1 + \omega_2$

$$\vec{P}_{NL} \rightarrow e_0\varepsilon_0\chi^{(2)}\left(2E_3\cdot E_2^* + 2E_3\cdot E_1^* + 2E_1\cdot E_2 + c.c.\right)$$

$$\Rightarrow \quad \frac{\partial E_1(z)}{\partial z} = \frac{i\mu_0\varepsilon_0\chi^{(2)}\omega_1^2}{k_1} E_3(z)\, E_2^*(z)\, e^{i(k_3-k_1-k_2)z}$$

$$\frac{\partial E_2(z)}{\partial z} = \frac{i\mu_0\varepsilon_0\chi^{(2)}\omega_2^2}{k_2} E_3(z)\, E_1^*(z)\, e^{i(k_3-k_1-k_2)z}$$

$$\frac{\partial E_1(z)}{\partial z} = \frac{i\mu_0\varepsilon_0\chi^{(2)}\omega_3^2}{k_3} E_1(z)\, E_2(z)\, e^{i(k_1+k_2-k_3)z}$$

We could let $k_3 - k_1 - k_2 = \Delta k$.

To simplify things, let $\Delta k = 0$. (assume phase match)

Also assume $E_3(z)$ is very strong, and $\frac{\partial E_3(z)}{\partial z} = 0$

$E_3(z) = E_3(0)$

$$\therefore \quad \frac{\partial E_1(z)}{\partial z} = i\,\frac{\mu_0 \varepsilon_0 \chi^{(2)} \omega_1^2}{k_1} E_3(0) \cdot E_2^*(z) \equiv K_1 \cdot E_2^*(z)$$

$$\frac{\partial E_2(z)}{\partial z} = \frac{i\,\mu_0 \varepsilon_0 \chi^{(2)} \omega_2^2}{k_2} E_3(0) \cdot E_1^*(z) \equiv K_2 E_1^*(z)$$

Standard method to solve it:

$$\frac{\partial^2 E_1(z)}{\partial z^2} = K_1 \frac{\partial}{\partial z} E_2^*(z) = K_1 \cdot K_2^* E_1(z)$$

$$= \frac{\mu_0^2 \varepsilon_0^2 \chi^{(2)2} \omega_1^2 \omega_2^2}{k_1^2 k_2^2} |E_3(0)|^2 \cdot E_1(z) \equiv \gamma^2 E_1(z).$$

The solution? $E_1(z) = A_1 e^{\gamma z} + B_1 e^{-\gamma z}$

Similarly we will have: $E_2(z) = A_2 e^{\gamma z} + B_2 e^{-\gamma z}$

Boundary condition: $E_2(z=0) = A_2 + B_2 = 0$; $A_2 = -B_2$.

$E_1(z=0) = E_1(0) = A_1 + B_1$

Another use of the boundary condition:

$$\frac{\partial E_1(z=0)}{\partial z} = K_1 E_2^*(z=0) = 0 \Rightarrow \gamma A_1 - \gamma B_1 = 0 \Rightarrow A_1 = B_1$$

$$\frac{\partial E_2(z=0)}{\partial z} = K_2 E_1^*(z) \Rightarrow \gamma (A_2 - B_2) = K_2 \cdot E_1^*(z=0)$$

;

Notice that we haven't choose our time frame yet.
We can select $E_3(z=0)$ to be real.
So $E_1(z=0)$ becomes a complex number, in general.

$$\gamma = \frac{\mu_0 \varepsilon_0 \chi^{(2)} \omega_1 \omega_2}{\sqrt{k_1 k_2}} E_3(0);$$

$$A_2 - B_2 = \frac{k_2 E_1^*(0)}{\gamma} = \frac{i\mu_0 \varepsilon_0 \chi^{(2)} \omega_2^2 E_3(0)}{k_2 \cdot \frac{\mu_0 \varepsilon_0 \chi^{(2)} \omega_1 \omega_2 E_3(0)}{\sqrt{k_1 k_2}}} \cdot E_1^*(0)$$

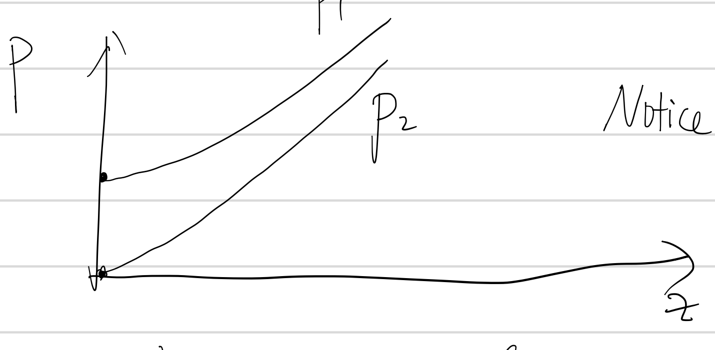
$$A_2 - B_2 = i \frac{\omega_2}{\omega_1} \sqrt{\frac{k_1}{k_2}} E_1^*(0)$$

$$= i \frac{\omega_2}{\omega_1} \sqrt{\frac{n_1 \omega_1}{n_2 \omega_2}} E_1^*(0) = i \sqrt{\frac{n_1 \omega_2}{n_2 \omega_1}} \cdot E_1^*(0).$$

$$\begin{cases} A_1 = B_1 \\ A_1 + B_1 = E_1(0) \end{cases} \Rightarrow E_1(z) = \frac{E_1(0)}{2}\left(e^{\gamma z} + e^{-\gamma z}\right) = E_1(0)\cosh(\gamma z)$$

$$\begin{cases} A_2 = -B_2 \\ A_2 - B_2 = i\sqrt{\frac{n_1 \omega_2}{n_2 \omega_1}} E_1^*(0) \end{cases} \Rightarrow E_2(z) = i\sqrt{\frac{n_1 \omega_2}{n_2 \omega_1}} \frac{E_1^*(0)}{2}\left(e^{\gamma z} - e^{-\gamma z}\right)$$

$$= i\sqrt{\frac{n_1 \omega_2}{n_2 \omega_1}} E_1^*(0) \sinh(\gamma z)$$



Notice the growth rate of P_1 and P_2 is not equal

The growth rate of photon number is equal.