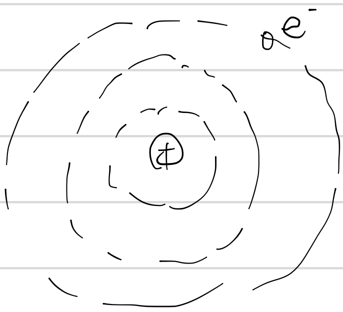
Revisit of Quantum Mechanics Postulates

Last two lectures, I was actually following standard "textbook" way of teaching postulates of quantum mechanics, and Apparently, it didn't work very well.

I will try my own style and revisit the postulates. I like to follow what really happened in history, because it explains why people came up with these postulates.

History Routes:

De Broglie proposed electron is wave, to solve the hydrogen atom spectral problem.



Postulate 1: system describe by a wavefunction. $\psi(x,t)$

$$\frac{1}{\lambda} = R_0 \left(\frac{1}{n^2} - \frac{1}{m^2}\right)$$

Schordinger: Wave must obey a wave-equation $\Leftrightarrow (F = ma)$

$$\psi = e^{ikx-i\omega t}, \quad p\psi = \frac{\hbar}{i}\frac{\partial}{\partial x} e^{ikx-i\omega t} = \frac{\hbar}{i}\frac{\partial}{\partial x}\psi = \hbar k \psi = \hat{p}\psi$$

Similarly, $$E\psi = i\hbar \frac{\partial}{\partial t} e^{ikx-i\omega t} = \hbar\omega\psi = i\hbar\frac{\partial}{\partial t}\psi.$$

And: $E = \frac{p^2}{2m} + V(x)$

$$E\psi = \hbar\omega\psi$$

$$\Downarrow$$

$$\left\{ \frac{\hat{p}^2}{2m} + V(\hat{x}) \right\} \psi(x,t) = i\frac{\partial}{\partial t}\psi(x,t)$$

Summarize two postulate:

Postulate 2: Wave equation satisfies Schordinger equation.

Postulate 3: physics attribute $\Leftrightarrow$ an operator.

Now, Schordinger needs to do something to show his equation is correct. So he tried to solve the Schordinger equation for Hydrogen atom. His equation needs to be tested by experiment.

$$\left\{ \frac{\hat{p}^2}{2m} - \frac{ke^2}{r} \right\} \psi(\vec{x},t) = i\frac{\partial}{\partial t}\psi(\vec{x},t)$$

$\hat{p} = \frac{\hbar}{i}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$ $\quad$ $\vec{p}$ is vector, so should $\hat{p}$.

but $\vec{p}^2$ is a number, so $\hat{p}^2 = \hat{p}\cdot\hat{p} = -\hbar^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)$

$$\left\{-\hbar^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)-\frac{ke^2}{\sqrt{\hat{x}^2+\hat{y}^2+\hat{z}^2}}\right\}\psi(\vec{x},t) = i\hbar\frac{\partial}{\partial x}\psi(\vec{x},t)$$

And.....

We are not going to solve it... Although Schordinger did solve it in history...

But how schordinger solved it is extremely interesting.

The left hand-side, $\hat{H}$ operator, is independent of time.
The right hand-side, $+i\hbar\frac{\partial}{\partial t}$, is independent of position.

So you can separate your solution $\psi(\vec{x},t)$ to

$$\psi(\vec{x},t) = \varphi_{\vec{x}}(\vec{x})\cdot\Phi_t(t)$$; separation of variables.

Now is the guessing around part..... You can let $\Phi_t = e^{-i\omega t}$.

The simplest solution.

$$\hat{H}(\vec{x})\cdot\varphi_x(\vec{x})\cdot e^{-i\omega t} = \hbar\omega\cdot\varphi_x(\vec{x})\cdot e^{-i\omega t}$$

$$\hat{H}(\vec{x})\,\varphi_x(\vec{x}) = \underset{\substack{\downarrow \\ E}}{\hbar\omega}\,\varphi_x(\vec{x})$$

Have you notice something? This is the definition of eigenvalue / eigenfunction.

$$\hat{A}\,\psi_a = a\cdot\psi_a$$

$\hat{A} \to \hat{H}$; $\psi_a \to \varphi_{\vec{x}}(\vec{x})$; $a \to \hbar\omega$, or E.

So the solution of Schordinger equation becomes a surprising mathematical problem.

Solve eigen-function of $\hat{H}\,\varphi_{\vec{x}}(\vec{x}) = E\,\varphi_{\vec{x}}(\vec{x})$

And the solution to Schordinger equation would be:

$$\psi(\vec{x},t) = \varphi_{\vec{x}}(\vec{x})\, e^{-i\frac{E}{\hbar}t} \quad ; \quad E = \hbar\omega.$$

So if you can solve this eigenfunction problem, you solved Schordinger equation.

So Schordinger solved this eigenfunctions, and he formed many solutions (or group of solutions)

with $E_n \propto \frac{R'}{n^2}$; n is integer.

So: $E_n - E_m = \hbar\omega \propto \frac{1}{\lambda} = R_0\left(\frac{1}{n^2} - \frac{1}{m^2}\right)$.

Reproduce the spectrum of Hydrogen Atom.

Schordinger solved the eigenvalues of $\hat{H}$ (energy operator), and the values are the same as experiment measurement of energy.

And in fact, all energy values extracted from the hydrogen spectra measurement, equals to the eigenvalues of $\hat{H}$!

There's no exception. This is a one to one mapping.

$$E_{measured} \leftrightarrow E_{eigenvalue}.$$

People summarise postulate 4:

The measurable value of attribute A, is the eigenvalues of operator $\hat{A}$.

It make sense ...

Schordinger's contribution stopped here.

Contime: From electron Young's double slit experiment: we know the linear combition of wave function should still be a valid wave function:

$$\psi = \sum_{n=1}^{\infty} \alpha_n \cdot \psi_n \qquad \text{Postulate 5.}$$

The final postulate (which Einstein never believed)

This one is extremely difficult to understand, but let's try.

Still Young's double Slit experiment of electrons.

We've seen the movie, electron arrives one by one, and people measures its position.

So according to postulate 4, all possible results are the eigenvalue of position operator $\hat{x}$.

And according to Postulate 5: all wavefunction can be expressed as superposition of eigenstates. $\psi(x) = \sum_{n=1}^{\infty} a_n \psi_n$.

So what is the eigenvalue and eigenfunction of $\hat{x}$?

eigenfunction is $\delta(x-x_0)$; eigenvalue is $\forall x_0$ (any).

Proof: $$\underset{\text{operator}}{\underset{\downarrow}{\hat{x}}}\,\psi(x) = \underset{\text{Variable}}{\underset{\downarrow}{x}}\,\psi(x) = \underset{\text{number}}{\underset{\downarrow}{x_0}}\,\psi(x) \Rightarrow (x-x_0)\psi(x) = 0.$$

Since x is a Variable, x_0 is a number, $x - x_0 \neq 0$ unless $x = x_0$.

So for $x \neq x_0$, $\psi(x) = 0$;

for $x = x_0$, $\psi(x = x_0)$ can be aritary value.

But since we need eigenfunction to be normalized:

$$1 = \int_{-\infty}^{+\infty} \psi(x)\, dx = \int_{x=x_0-\varepsilon}^{x=x_0+\varepsilon} \psi(x)\, dx \quad ; \text{ As } \varepsilon \to 0;\ \psi(x) \to \infty.$$

So we have: the eigenfunction of $\hat{x}$ for eigenvalue of x_0 is:

$$\psi(x) = \begin{cases} 0, & x \neq x_0 \\ \infty & x = x_0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{+\infty} \psi(x)\, dx = 1$$

This is the definition of δ-function: $\psi(x) = \delta(x - x_0)$.

So $\hat{x}$ has eigenvalue x_0, eigenfunction $\delta(x - x_0)$, x_0 can be any real number.

Now let's try it on Double slit experiment. From that experiment, we know individual electrons arrives randomly, but its probability of arriving at position x_0, is $|\psi(x_0)|^2$ if $\psi(x_0) = e^{ikr_1} + e^{ikr_2}$

$$r_1 = \sqrt{L^2 + (x_0 - \tfrac{d}{2})^2} \quad ; \quad r_2 = \sqrt{L^2 + (x_0 + \tfrac{d^2}{2})}.$$

Exactly the same as light.

$$\underset{\substack{\downarrow \\ \text{probability}}}{P(x_0)} = |\psi(x_0)|^2.$$

Now let's combine with postulate 4 and 5.

$\psi(x)$ can be expressed as supposition of eigenstates of $\hat{x}$: $\delta(x-x_0)$

$$\psi(x) = \sum_{\forall x_0} \alpha(x_0) \cdot \delta(x-x_0) = \int_{-\infty}^{+\infty} \alpha(x_0)\, \delta(x-x_0)\, dx_0$$

$$= \alpha(x).$$

$$\therefore \psi(x) = \int_{-\infty}^{+\infty} \underline{\psi(x)} \cdot \underline{\delta(x-x_0)}\, dx_0.$$

And we know the probability of measuring x, and get result x_0 is:

$$|\psi(x_0)|^2$$

We express $\psi(x)$ as the superposition of the eigenfunctions of $\hat{x}$, then when we measure $\hat{x}$, the probability of getting x_0 eigenvalue is the amplitude square of the coefficient in front of the eigenstate.

Extend it to all operators.

Let $\hat{x} \to \hat{A}$; $\delta(x-x_0) \to \psi_{a,n}$; $x_0 \to a_n$

We have Postulate 6:

When measuring A on state $\psi = \alpha_n \psi_{a,n}$, when $\psi_{a,n}$ is the n-th eigenfunction of operator $\hat{A}$, then the probability of getting eigenvalue a_n, is $|\alpha_n|^2$.