Week 2

Wave equation from Maxwell equation

Start with : $\rho_{free} = 0$; $\vec{J} = 0$.

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t}$$

Standard procedue:

$$\nabla \times (\nabla \times \vec{E}) = -\frac{\partial}{\partial t}(\nabla \times \vec{B}) = -\mu_0 \frac{\partial^2 \vec{D}}{\partial t^2}.$$

Let's have $\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} + \vec{P}_{NL}$

RHS: $= -\mu_0 \varepsilon_0 \varepsilon_r \cdot \frac{\partial^2 \vec{E}}{\partial t^2} - \mu_0 \frac{\partial^2 \vec{P}_{NL}}{\partial t^2}$

LHS: Use a math equation:

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}.$$

$$\nabla = \hat{e}_x \cdot \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\nabla^2 \cdot \vec{E} = (\nabla \cdot \nabla)\vec{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\vec{E}.$$

$$\nabla(\nabla \cdot \vec{E}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}E_x + \frac{\partial}{\partial y}E_y + \frac{\partial}{\partial z}E_z\right)$$

$\nabla\cdot\vec{E} = 0.$ True for transverse plane wave $\vec{E}e^{-i\omega t+ikz}$;

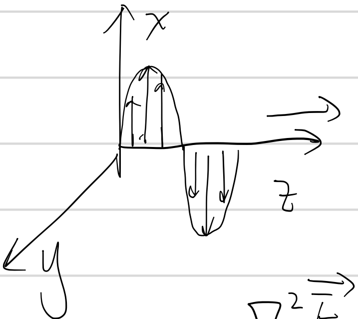
And also true for the sum of plane waves.

☆ One can proof that $\nabla(\nabla\cdot\vec{E})$ is much smaller than all other term is the equation.

Therefore, we have:

$$\nabla^2\vec{E} - \mu_0\varepsilon_0\varepsilon_r\frac{\partial^2\vec{E}}{\partial t^2} = \mu_0\frac{\partial^2 \vec{P}_{NL}}{\partial t^2} \qquad (*)$$

In Linear regime, $\vec{P}_{NL}=0$, Start with easiest wave: Transverse wave:



Oscillation direction is perpendicular to the propagation direction!

$$\vec{E} = (E(z), 0, 0)$$

$$\nabla^2\vec{E} = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\vec{E} = \frac{\partial^2\vec{E}}{\partial z^2}$$

Reduce equation to:

$$\frac{\partial^2\vec{E}}{\partial z^2} - \mu_0\varepsilon_0\varepsilon_r\frac{\partial^2\vec{E}}{\partial t^2} = 0.$$

Standard wave-equation. Solution is $\vec{E} = \hat{e}_x E_0\cos(\omega t - kz)$

Where $\frac{\omega}{2\pi}$ is the frequency, k is the wave-vector defined as

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{c_r\cdot T} = \frac{2\pi}{c_r\cdot\frac{1}{f}} = \frac{2\pi f}{c_r} = \frac{\omega}{c_r}$$

plug this back, using:

$$\frac{\partial}{\partial x}\frac{\partial}{\partial x}\cos(ax) = \frac{\partial}{\partial x}(-a\sin ax) = -a^2\cos ax.$$

$$-k^2\vec{E} + \mu_0\varepsilon_0\varepsilon_r\cdot\omega^2\cdot\vec{E} = 0.$$

$$\therefore \quad k = \sqrt{\mu_0\varepsilon_0\varepsilon_r}\,\omega = \frac{\omega}{C_r}$$

$$\therefore \quad C_r = \frac{1}{\sqrt{\mu_0\varepsilon_0\varepsilon_r}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\cdot\frac{1}{\sqrt{\varepsilon_r}} = \frac{c}{n};$$

$$\therefore \quad \varepsilon_r = n^2.$$

Important to remember,

$k^2 = \mu_0\varepsilon_0\varepsilon_r\omega^2$; we will keep using this result to simplify equations in this class.

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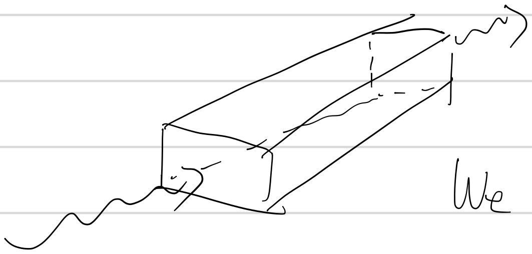
Nonlinear Regime



Given $\vec{E}(z=0)$, Solve $\vec{E}(z=L)$

We will have plane wave again, but:

$\vec{E} = \hat{e}_x E(z) e^{-i\omega t + ikz} + c.c.$; we only calculate one part, and add c.c.

c.c. : Complex conjugate.

$$\nabla^2 \vec{E} = \frac{\partial^2}{\partial z^2}\vec{E} = \hat{e}_x \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}E(z) \cdot e^{-i\omega t + ikz} + ikE(z)e^{-i\omega t + ikz}\right) + c.c.$$

$$= \Big(\frac{\partial^2}{\partial z^2}E(z)e^{-i\omega t+ikz} + ik\frac{\partial}{\partial z}E(z) \cdot e^{-i\omega t+ikz}$$

$$+ ik\frac{\partial}{\partial z}E(z) \cdot e^{-i\omega t+ikz} - k^2 E(z)e^{-i\omega t+ikz} + c.c. \Big)\hat{e}_x$$

LHS: $\nabla^2 \vec{E} - \mu_0\varepsilon_0\varepsilon_r \frac{\partial^2}{\partial t^2}\vec{E}$

$$= \hat{e}_x \Big\{ -k^2 \underbrace{E(z)e^{-i\omega t+ikz} + \mu_0\varepsilon_0\varepsilon_r\omega^2 E(z)e^{-i\omega t+ikz}}_{=0}$$

$$+ e^{-i\omega t+ikz}\frac{\partial^2}{\partial z^2}E(z) + 2ike^{-i\omega t+ikz}\frac{\partial}{\partial z}E(z) \Big\}. + c.c.$$

LHS: $$= e^{-i\omega t+ikz}\left\{ \frac{\partial^2}{\partial z^2}E(z) + 2ik\frac{\partial}{\partial z}E(z) \right\} + c.c.$$

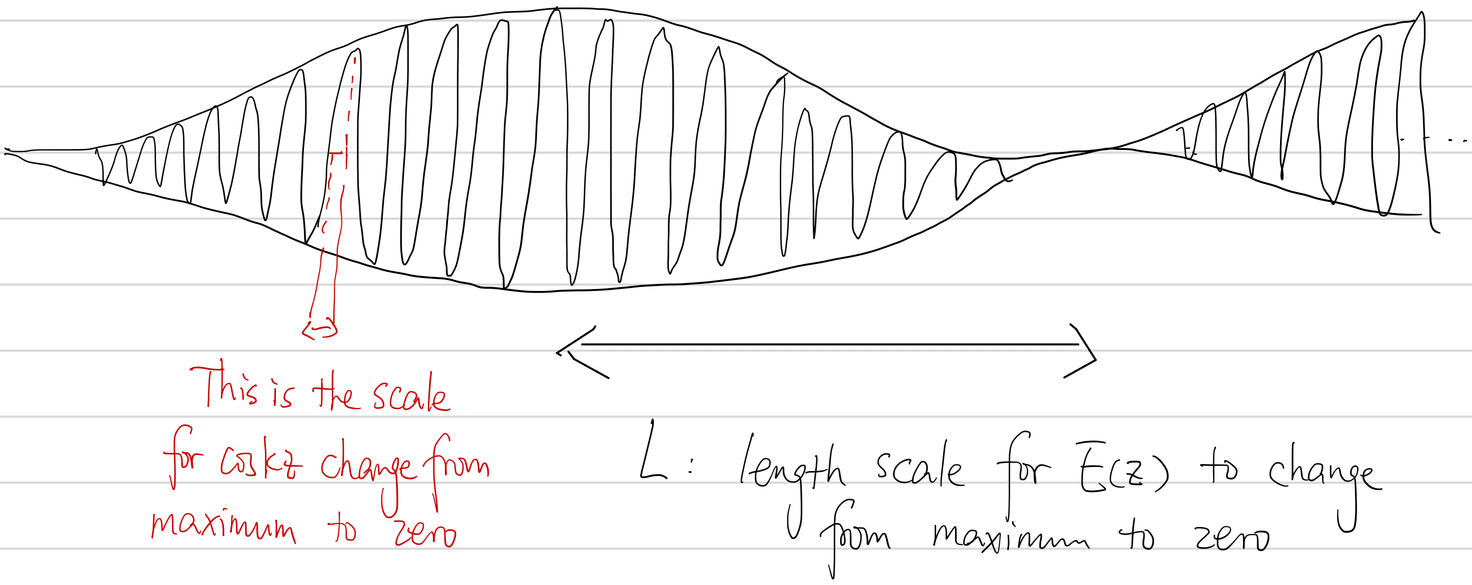
"Slow varying amplitude approximation"

Question, $\frac{\partial^2 E(z)}{\partial z^2}$ vs. $k\left(\frac{\partial E(z)}{\partial z}\right)$, which is larger?

$E = E(z)\cos(\omega t - kz)$

envelope wave.

At any given time t, a regular wave would look like this:



$\therefore \quad k \longleftrightarrow \frac{\partial}{\partial z}\cos(\omega t - kz)$

$\frac{1}{L} \longleftrightarrow \frac{\partial}{\partial z}E(z).$

Approximation: if the amplitude variation is much slower than the wave oscillation, then: $\frac{\partial^2 E(z)}{\partial z^2} \ll k\,\frac{\partial E(z)}{\partial z}$.

Apply this approximation to LHS:

$$LHS = 2ik\, e^{-i\omega t + ikz} \frac{\partial E(z)}{\partial z} + c.c.$$

If there are multiple frequency components, then:

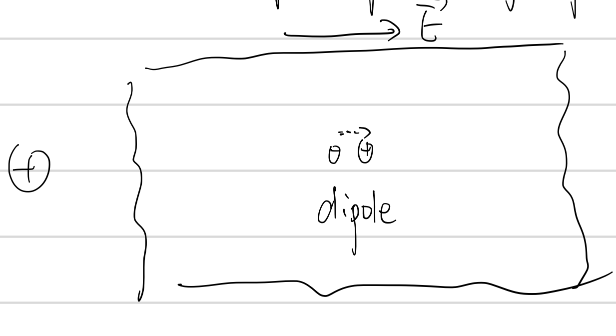
$$E(t,z) = \sum_j E_j(z) e^{-i\omega_j t + ik_j z} + c.c.$$

$$LHS = 2i \sum_j k_j e^{-i\omega_j t + ik_j z} \frac{\partial E_j(z)}{\partial z} + c.c.$$

$$RHS: \quad = \mu_0 \frac{\partial^2 \vec{P}_{NL}}{\partial t^2}$$

Question: what is $\vec{P}_{NL}$?

Microscopic point of $\vec{P}_{NL}$.



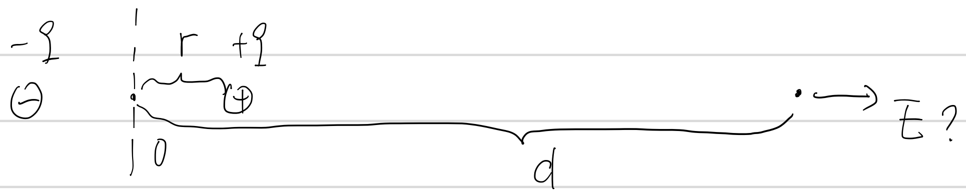
$$\nabla \cdot \vec{P} = -\rho'$$

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} + \vec{P}.$$

Unit of $\vec{P}$: $\varepsilon_0 \vec{E} \sim \varepsilon_0 \cdot \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \sim \frac{q}{L^2}$

What dipole can do? $\vec{P} = N \cdot \langle q \cdot \vec{r} \rangle$

Keep in mind: $\vec{P}$ is a "field" induced by dipoles.



In molecule, or atom, so r is very small, smaller than nm. (the size of atom / molecule)

Calculate E generated by dipole at the distance

$d \gg r$.

$$E_d = \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{(d-r)^2} - \frac{1}{(d+r)^2}\right\}$$

Use approximation: $(1+x)^y \longrightarrow 1 + y \cdot x$, if $|x| \ll 1$.

$$E_d = \frac{q}{4\pi\varepsilon_0 d^2}\left\{\left(1-\frac{r}{d}\right)^{-2} - \left(1+\frac{r}{d}\right)^{-2}\right\}$$

$$= \frac{qr}{\pi\varepsilon_0 d^3} \quad \propto \frac{1}{d^3}\cdot(q\cdot r)$$

"Field" Generated by dipole.

Macroscopic point of view $\vec{P} \propto N \cdot (q \cdot \vec{r})$

N: number density

More detailed calculation shows:

$\vec{P} = N \cdot q\vec{r}$; Check Unit: $N \cdot q \cdot \vec{r} = \frac{1}{L^3} \cdot q \cdot L = \left[\frac{q}{L^2}\right]$

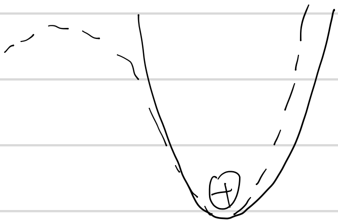
Correct.

$\vec{P} = N \cdot q \cdot \vec{r}$

$\vec{P}_{LN} = (\varepsilon_r - 1)\vec{E}$; In linear regime, $\vec{r} \propto \vec{E}$.

What would nonlinear $\vec{P}$ looks like?

Go back to a classical picture:
a charge in potential well.



$$V(x) = \frac{1}{2}kx^2 + \frac{1}{3}\alpha k x^3 + \dots \quad ; \quad \alpha \ll 1$$

Workflow: System driven by $\vec{E}$, calculation motion of the charge $r(\vec{E})$ then obtain $\vec{P}_{NL}$.

Force: $F = -\frac{dV(x)}{dt} = -kx - \alpha x^2$

Because this is a very small motion, $\langle x \rangle \ll \lambda$, so
$\vec{E}(x,t) = \vec{E}(t) = \sum_i E_i e^{-i\omega_i t} + c.c.$

$$m \cdot \ddot{x} = -kx - \alpha x^2 + qE - \gamma \frac{dx}{dt}$$

("air drag")
damping: viscous force in harmonic oscillator.

$$m\ddot{x} + \gamma\dot{x} + kx + \alpha x^2 = qE(t).$$

Let $\alpha = 0$, approach linear regime:

$$\ddot{x} + \frac{\gamma}{m}\dot{x} + \frac{k}{m}x = \frac{q}{m}E_1 e^{-i\omega_1 t} \quad ; \quad \text{Notice: } \omega_0 = \sqrt{\frac{k}{m}}.$$

Solution: $x = x_1(t)\, e^{-i\omega_1 t}$; Steady state $x_1(t) \to x_1$

$$\Rightarrow \quad -\omega_1^2 x_1 - \frac{i\omega_1\gamma}{m} x_1 + \omega_0^2 x = qE_1/m$$

$$x_1 = \frac{qE_1/m}{(\omega_0^2 - \omega_1^2) - \frac{i\omega_1\gamma}{m}}.$$

$$\vec{P} = \langle N\rangle \cdot q \cdot r = \langle N\rangle \cdot q \cdot x_1 = \frac{\langle N\rangle \cdot q^2/m}{(\omega_0^2 - \omega_1^2) - \frac{i\gamma\omega_1}{m}} \cdot \vec{E}$$

If $\gamma = 0$, then $\vec{P} = \varepsilon_0 \chi \vec{E}$, χ is real number.

$\gamma \neq 0$, then imaginary number, what does it mean?
"Absorption".

$$\vec{D} = \varepsilon_0\varepsilon_r \vec{E} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0(1+\chi)\vec{E} \ ; \quad \varepsilon_r = 1+\chi = n^2$$

$n = \sqrt{1+\chi}$; $\gamma \neq 0$ means imaginary part of n is not zero.

Take a plane wave as example: $E(z) = E e^{-i\omega t + ikz}$;

$$k = \frac{2\pi n}{\lambda} = \frac{2\pi n_{Real} + 2\pi i n_{Im}}{\lambda} \ ; \quad E(z) = E e^{-i\omega t + i k_{real} z} \cdot \underbrace{e^{-\frac{2\pi n_{Im}}{\lambda} z}}_{\text{Loss.}}$$

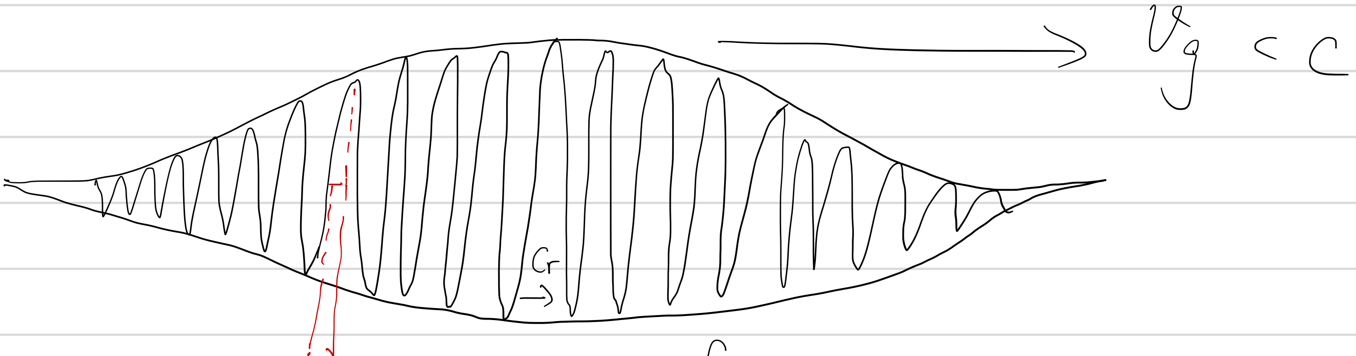
Another observation that doesn't make much sense at the first look: let $\gamma \to 0$, then if $\omega_1 > \omega_0$,

$$\chi < 0 ; \quad \to \quad n < 1, \quad c_r = \frac{c}{n} > c$$

Faster than speed of light ?

The speed here, is called phase velocity.
Another speed, is group velocity (envelope velocity)

$n<1$ means: phase is moving forward faster than light, but as long as the envolope is not, no information/energy is transmitted faster than light



$c_r > c$, fine.

This the classical model of Linear Optics.

Nonlinear Polarization:

$$m\ddot{x} + \gamma\dot{x} + kx + \alpha k x^2 = qE(t)$$

Notice $\alpha k x^2$ is perturbation to the equation, such that:
$\alpha k x^2 << kx$. ; $\alpha << 1$

Perturbation Method. ; Talyor Expand $x(t)$,

$$x(t) = x_1 e^{-i\omega_1' t} + \alpha \cdot x_2 e^{-i\omega_2' t} + \alpha^2 x_3 e^{-i\omega_3' t} + \dots$$

Plug back to equation:

$$\left(m\frac{d^2}{dt^2} + \gamma\frac{d}{dt} + k\right)\left(x_1 e^{-i\omega_1' t} + \alpha x_2 e^{-i\omega_2' t} + \dots\right)$$
$$+ \alpha k x_1^2 e^{-2i\omega_1' t} + \alpha^3 k x_2^2 e^{-2i\omega_2' t} \dots = qE e^{-i\omega_1 t};$$

Collect terms based on the order of α.

α^0: $\left(m\frac{d^2}{dt^2} + \gamma\frac{d}{dt} + k\right) x_1 e^{-i\omega_1' t} = qE e^{-i\omega_1 t}$; (1)

Solution is the same as $\alpha = 0$ (shown previously)

α^1: $\alpha\left(m\frac{d^2}{dt^2} + \gamma\frac{d}{dt} + k\right) x_2 e^{-i\omega_2' t} = -\alpha k x_1^2 e^{-2i\omega_1' t}$; (2)

....

Solution: $x_1 = \dfrac{qE_1/m}{(\omega_0^2 - \omega_1^2) - \frac{i\omega_1\gamma}{m}}$

Equation α' is identical form of α^0, but

$qE e^{-i\omega_1 t}$ replaced by $-kx_1^2 e^{-2i\omega_1 t}$

$$\Rightarrow \quad X_2 = \frac{kX_1^2/m}{(\omega_0^2 - 4\omega_1^2) - \frac{2i\omega_1\gamma}{m}} \quad , \text{ and } \omega_2' = 2\omega_1$$

Interesting to plug in X_1

$$x_2(t) = X_2 e^{-i\omega_2' t} = \frac{kq^2/m^3 \cdot E_1^2 \, e^{-2i\omega_1 t}}{[(\omega_0^2 - \omega_1^2) - \frac{i\gamma\omega_1}{m}]^2 [(\omega_0^2 - 4\omega_1^2) - \frac{2i\omega_1\gamma}{m}}$$

$$\propto \left(E_1 e^{-i\omega_1 t}\right)^2 \quad ; \quad P = \langle N \rangle \cdot q x$$

$$P = P_1 + P_2 + \ldots$$

$$\propto \chi^{(1)} E(t) + \chi^{(2)} E^2(t) + \ldots\ldots$$

This make sense, because if we can Taylor expand x. to αx, $\alpha^2 x$, $\alpha^3 x$ .....

We should also be able to Tylor Expand $P(E)$,

to E, E^2, E^3 ......

As long as $\chi^{(1)} E \gg \chi^{(2)} E^2 \gg \chi^{(3)} E^3 \gg \ldots\ldots$

So definition (Tylor expansion)

$$\vec{P} = \underbrace{\varepsilon_0 \chi^{(1)} \vec{E}}_{\vec{P}_{LN}} + \underbrace{\varepsilon_0 \chi^{(2)} \vec{E}\vec{E} + \varepsilon_0 \chi^{(3)} \vec{E}\vec{E}\vec{E} + \cdots\cdots}_{\vec{P}_{NL}}.$$

General Framework of Light and matter interaction in perturbation regime:

$$\boxed{\begin{gathered} \sum_j 2ik_j \frac{\partial E_j}{\partial z} e^{-i\omega_j t + ik_j z} + c.c. = \mu_0 \frac{\partial^2 \vec{P}_{NL}}{\partial t^2} \\ \vec{P}_{NL} = \varepsilon_0 \vec{\vec{\vec{\chi}}}^{(2)} \cdot \vec{E}\vec{E} + \varepsilon_0 \vec{\vec{\chi}}^{(3)} \cdot \vec{E}\vec{E}\vec{E} + \cdots\cdots \end{gathered}}$$

This is true for all materials in perturbation regime,

Different materials will have different nonlinear coefficient.

Notice $\vec{P}_{NL}$ is a vector, but $(\vec{E})^2$ is a number, so we are defining χ in a special mathematical way:

$\vec{\vec{\chi}}$ is a tensor. $\vec{\vec{\chi}}^{(2)}$ is a rank-three tensor, everytime a tensor multiply with a vector, its rank reduce one. And rank-1 tensor is a vector; rank-2 tensor is a matrix.

$\therefore$ $\vec{\vec{\vec{\chi}}}^{(n)}$ is $(n+1)$-rank tensor, and $\vec{\vec{\vec{\chi}}}^{(n)} \underbrace{\vec{E}\vec{E}\cdots\vec{E}}_{n}$ is a vector.

A quick example: for linear polarization, we have

$\vec{P}_{NL} = \varepsilon_0 \overleftrightarrow{\chi}^{(1)} \cdot \vec{E}$; $\overleftrightarrow{\chi}$ is rank-2 tensor, so it can be represented as a matrix.

For most of the material, we can write:

$$\overleftrightarrow{\chi} \cdot \vec{E} = \begin{pmatrix} \chi_x & 0 & 0 \\ 0 & \chi_y & 0 \\ 0 & 0 & \chi_z \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

For isotropic material, $\chi_x = \chi_y = \chi_z$; $\overleftrightarrow{\chi}$ reduce to $\chi \cdot \overleftrightarrow{I} = \chi$. a number times identity matrix.

However, for anisotropic material, such as crystal, χ_x, χ_y, χ_z can be different.

This is the mathematical way of describing "refration index" dependence on the orientation.

Example: Birefringence effect.

A quick peek of nonlinear phenomena.

$$\sum 2ik_j \frac{\partial E_j}{\partial z} \cdot e^{-i\omega_j t + ik_j z} + c.c. = \mu_0 \frac{\partial^2}{\partial t^2} \vec{P}_{NL}$$

$$\vec{P}_{NL} = \varepsilon_0 \overleftrightarrow{\chi}^{(2)} \vec{E}\vec{E} + \varepsilon_0 \overleftrightarrow{\chi}^{(3)} \vec{E}\vec{E}\vec{E} + \ldots$$

1. Frequency doubling and tripling ......

Neglect the tensor complication first.

If we have input of $E_0 e^{-i\omega_0 t + ik_0 z} + c.c.$

In P_{NL}, we have $(E_0 e^{-i\omega_0 t + ik_0 z} + c.c. + \ldots)^2$
$= E_0^2 e^{-2i\omega_0 t + 2ik_0 z} + c.c. + \ldots\ldots$

A dipole $\vec{P}$ oscillating at $2\omega_0$, with wave vector of $2k_0$.
so an electrical field at this frequency can be generated.

Similarly, $\vec{E}^3$ term gives you $3\omega_0$.

Common application: Get laser wavelength to the regime where laser generation is either technical challenging or commercially challenging.

$1.06\mu m \xrightarrow{\times 2} 532nm$, Green laser pointer

Near infered pulse laser $\xrightarrow{\times N \geq 20}$ UV laser.

2. Sum/ Difference frequency:

Input: $E_1 e^{-i\omega_1 t + ik_1 z} + E_2 e^{-i\omega_2 t + ik_2 z} + c.c.$

$\vec{P}_{NL} \propto (E_1 e^{-\cdots} + E_2 e^{-\cdots} + c.c.)^2$

contain terms: $E_1 E_2 e^{-i(\omega_1+\omega_2)t + i(k_1+k_2)z} + c.c.$

$+ E_1 E_2^* e^{-i(\omega_1-\omega_2)t + i(k_1-k_2)z} + c.c.$

$\omega_1 + \omega_2$: Sum frequency generation.

$\omega_1 - \omega_2$: Difference frequency generation.

☆ Special case in sum/difference frequency generation.

$\omega_2 \to 0$, E_2 is a static electrical field.

$\vec{P}_{NL}$: $\underbrace{\chi^{(2)} E_2}_{\text{a number}} \cdot E_1 e^{-i\omega_1 t + ik_1 z} + c.c.$

Let's compare with the linear polarization:

$\vec{P}_{LN}$: $\chi^{(1)} \cdot E_1 e^{-i\omega_1 t + ik_1 z} + c.c.$

And we know: $\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 (1+\chi^{(1)}) \vec{E} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon_0 n^2 \vec{E}$.

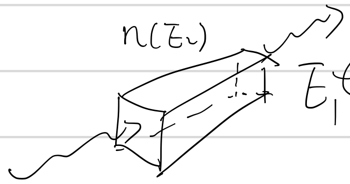
$\therefore n^2 = 1 + \chi^{(1)}$

Now $\vec{P} = \vec{P}_{LN} + \vec{P}_{NL} = \varepsilon_0 (\chi^{(1)} + \chi^{(2)} E_2) \vec{E}_1$

$\therefore \quad \varepsilon_r = 1 + \chi^{(1)} + \chi^{(2)} E_2 = n^2.$

The electrical field can vary refraction index!

This is the foundation of electro-optics modulator.



$E_1 e^{ikz} = e^{\frac{i 2\pi n(E_2) z}{\lambda}}$; phase of the output light depends on the electrical field.

E_1

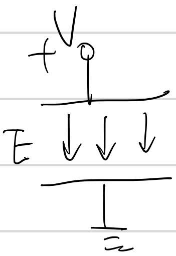
A slight change in $n(E_2)$ can leads to a large phase change.

For example, if $n(E_2) - n = 0.001$, $\lambda = 1\mu m$; $L = 1mm$.

then we have $\Delta\varphi = \frac{2\pi n(E_2) L}{\lambda} - \frac{2\pi n L}{\lambda} = \frac{2\pi L \cdot \Delta n}{\lambda} = 2\pi \times 1000 \times 0.001$
$= 2\pi.$

0.1% percent change in refraction index $\rightarrow$ 2π phase change.

In EO-phase modulator, a critical parameter is:
the Voltage (E_2) required to make a π-phase change.



V_π ; Currently, the leading company in this competition is EOspace in Washiton State,

$V_\pi < 3V$; You can change the phase with "battery"

Phase modulator / Intensity modulator is the most common method to convert information between optics and electronics. Widely used in telecommunication.

3. Third-order nonlinearity.

3.1 $\vec{P}_{NL} \propto \vec{E}^3$

$$\vec{E} = E_1 e^{-i\omega_1 t + ik_1 z} + E_2 e^{-i\omega_2 t + ik_2 z} + E_3 e^{-i\omega_3 t + ikz} + c.c.$$

Again, we will have sum frequency and harmonic generation, so we are not going to repeat here.

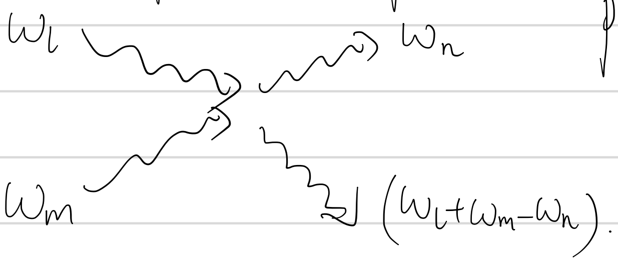
Four-wave mixing effect.

$$\vec{E}^3 = (+ \dots + \dots)(+ \dots + \dots)(+ \dots + \dots)$$

$$= \sum_{l,m,n} E_l \cdot E_m E_n^* e^{i(\omega_l + \omega_m - \omega_n)t + i(k_l + k_m - k_n)z} + c.c.$$

Notice, if ω_l, ω_m, ω_n are close in frequency, frequency $(\omega_l + \omega_m - \omega_n)$ will also be very close with them.

This process correspond to a two-photon to two-photon process, and we will discuss this in detail.



Elastic photon scattering

3.2 Special case: Now, we can write down

E as: $E = E_+ + c.c.$

where E_+ is $\sum_j E_j e^{-i\omega_j t + ik_j z}$; $E_+^* = \sum_j E_j^* e^{+i\omega_j t - ik_j z}$;

So: $E^3 = (E_+ + E_+^*)^3$

it contains a term:

$= E_+ \cdot E_+^* \cdot E_+ = |E_+|^2 E_+ + c.c.$

This is again very interesting:

$\vec{P}_{NL}$ has a term: $\varepsilon_0 \chi^{(3)} |E_+|^2 E_+ = \vec{P}_+'$

And from previous derivation, we know:

$$\vec{D}_+ = \underbrace{\varepsilon_0 \varepsilon_r \vec{E}_+ = \varepsilon_0 \vec{E}_+ + \vec{P}_+}\; = \varepsilon_0 (\underbrace{1 + \chi^{(1)} + \chi^{(3)} |E_+|^2}) \vec{E}_+ = \varepsilon_0 n^2 \vec{E}_+$$

So: $n = \sqrt{1 + \chi^{(1)} + \chi^{(3)} |E_+|^2} = \sqrt{n_0^2 + \chi^{(3)} |E_+|^2}$

where n_0 is the original refraction index,

We work in the regime where $\chi^{(3)} |E_+|^2 \ll n_0^2$;

So: $n = n_0 \left(1 + \frac{\chi^{(3)} |E_+|^2}{n_0^2}\right)^{1/2} \approx n_0 \left(1 + \frac{\chi^{(3)}}{2n_0^2} |E_+|^2\right)$

So refraction index is proportional to the field^2 of light.

From EM-field class, we know: $|E_+|^2 \propto I$,

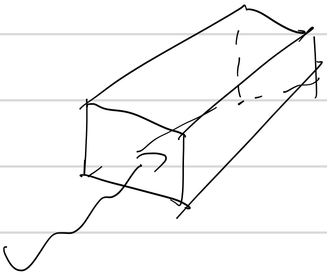
I is the intensity of light. unit: W/m^2

So we can write $n = n_0 + n_2 I$;
This is called Kerr effect.

Refraction index increases with the intensity of light.

Notice Kerr effect use term $|E_t|^2 E_t$,
it already includes terms in Four-wave mixing.
In later chapter we can show how to convert them from one to another.

Kerr effect is widely used in fetmosecond laser generation, and we will talk about this in later c .



$$E e^{-i\omega t + ikz} = E e^{-i\omega t + i\frac{2\pi n}{\lambda} z}$$

$$= E e^{-i\omega t + i\frac{2\pi n_0}{\lambda} z} \cdot e^{i\frac{2\pi n_0 I}{\lambda} z}$$

phase change.

So Kerr effect can induce a phenomena called:
"Self-phase modulation"

Kerr Effect includes $\begin{cases} \text{Four-wave mixing} \\ \text{self-phase modulation} \\ \text{Cross-phase modulation} \end{cases}$

3.3. In 3.2, we actual assume $\chi^{(3)}$ is real, so that n_2 is also real.

Now what happen if $\chi^{(3)}$ is a complex number?

Then $n_2 \to n'$, complex number.

$$n = n_0 + n'I = n_0 + Re(n')I + i\,Im(n')I.$$

$$E e^{-i\omega t + ikz} = E e^{-i\omega t + i\frac{2\pi n}{\lambda} z} =$$

$$E e^{-i\omega t + i\frac{2\pi Re(n')}{\lambda} I z} \cdot e^{-\frac{2\pi Im(n')}{\lambda} \cdot I \cdot z}$$

loss or gain.

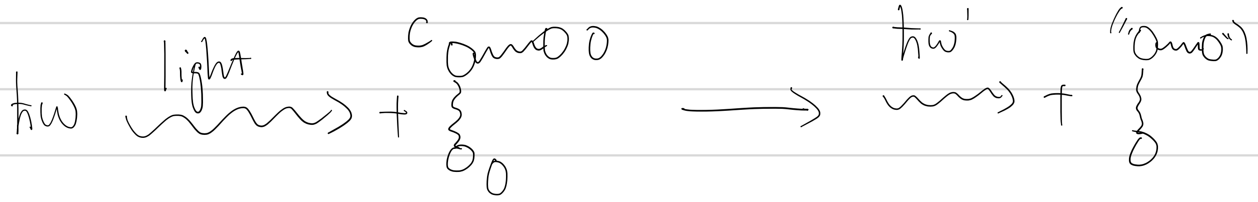
If $Im(n') < 0$, $E e^{\cdots} \nearrow$ with z, the light is gaining energy. ; $Im(n') > 0$, loss energy

But where does the energy comes from? Where does the energy go?

This is a regime of inelastic scattering, where the total energy in light will change.

There are two common process:

a. Raman scattering



CO_2 molecule

Oscillating molecule.

Give energy to molecule Vibration state.
The reverse process gain energy from molecule.

b. Brillioun scattering

solid

Vibrating solid.
"phonon"

Give energy to phonon in the material.

Reverse: gain ....