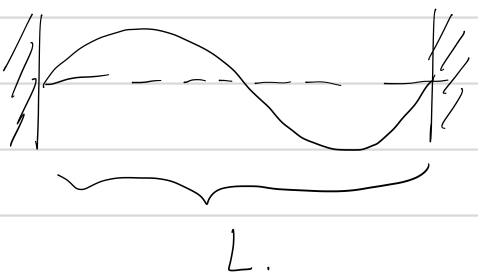
Week 4.

A critical step forward is accomplished by: De Broglie in 1924.

"Anything can be a wave"

What are the things that have discrete solutions?
Best example is wave with boundary condition



$N \cdot \lambda = L.$

"Wave + boundry condition"
Gives discrete solution.

Be Broglie is kind of an "outsider" for physics,
he had a B.A. for history in 1910.
science in 1913.
... phD .... in 1924.

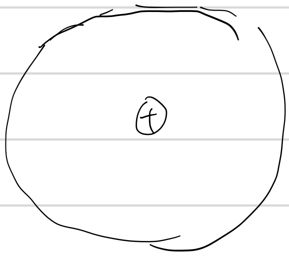
And the wavelength can be written as: (for light)

$\hbar k = p$: momentum.

$k = \frac{2\pi}{\lambda}$; $\therefore \frac{\hbar \cdot 2\pi}{\lambda} = \frac{h}{\lambda} = p$; $\lambda = \frac{h}{p}$

wave quanta $\begin{cases} \text{energy is: } \hbar\omega \\ \text{momentum is: } \hbar k \end{cases}$: photon (Einstein) $\longrightarrow$ All particle (De Broglie)

And if electron satisfied periodic boundary condition:



$$\boxed{2\pi r = n\lambda;}$$

And put it with classical model, you get the correct spectrum.

Debye made an important remark to his "post-doc" Schordinger when reading De Broglie's paper:

"If particles behave like waves, they must satisfy a wave equation".

See how this can work.

In classical physics, parallel to $F = ma$, are Lagragian and Hamiltonian mechanism.

Simply put: Kitnetic energy and potential energy gives the full dynamics of the system.

It's equivilent to $F = ma$.

$H = T + V$, describe the system

Wave: $e^{-i\omega t}$; what is its energy?

$\hbar\omega$, what is H? Energy.

$$i\hbar \frac{\partial}{\partial t} e^{-i\omega t} = \hbar\omega e^{-i\omega t}$$

$$\downarrow$$

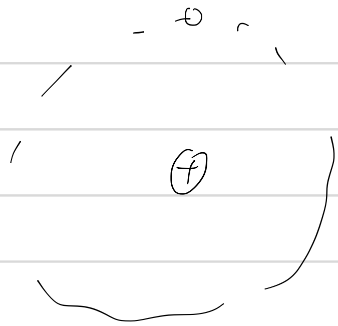
$$i\hbar \frac{\partial}{\partial t} |\psi\rangle \rightarrow E \cdot |\psi\rangle$$

$$\downarrow$$

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle$$

This is the Schordinger Equation.

For example, to solve Hydrogen Atom:



$$H = \frac{1}{2} m v^2 - \frac{e^2}{4\pi\varepsilon_0} \frac{1}{r}$$

For wave, we like momentum, instead of speed, so:

$$H = \frac{p^2}{2m} - \frac{e^2}{4\pi\varepsilon_0} \frac{1}{r}.$$

What is p? e^{ikx} is wave momentum part, $\hbar k = p$,

So: $-i\hbar \frac{\partial}{\partial x} e^{ikx} = \hbar k e^{ikx} = p e^{ikx}$

So a reasonable guess is : $\hat{p} = -i\hbar\nabla$

Now: $H = -\frac{\hbar^2}{m}\nabla^2 - \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r}$;

$H\psi = i\frac{\partial}{\partial t}\psi$

$\psi(\theta) = \psi(\theta + 2\pi)$

And the solution set : $\psi_n = \phi_n(x, y, z)\, e^{-i\omega_n t}$

gives $\omega_n \propto \frac{e^2}{n^2}$

So emission : $\omega = \omega_n - \omega_m \rightarrow \left(\frac{1}{n^2} - \frac{1}{m^2}\right)$

Perfect.

A few more remark:

ψ is probability wave, so: $\int \psi^* \psi \, d\vec{r} = 1$

Eigenwave: $H\psi_n = E_n \psi_n$

To calculate a quantity: $\langle E \rangle = \int \psi^* H \psi \, d\vec{r}$
$= \langle \psi | H | \psi \rangle$.

And $\psi = a_1\psi_1 + a_2\psi_2 + \dots$

$= \sum_n a_n \psi_n$; where $H\psi_n = E_n \psi_n$.

Orthogonal: $\int \psi_m^* \cdot \psi_n \, dx = \delta_{m,n}$.

Completeness:

The entire set of eigenstates (or eigenvector) forms a complete vector space for H.

Any $\psi(t)$ that is the solution to

$H|\psi(t)\rangle = i\frac{\partial}{\partial t}\psi(t)$

can be written as: $\psi(t) = \sum_n a_n(t)\,\psi_n(t)$

☆ Dirac notation: $\psi \to |\psi\rangle$; $\psi^* \to \langle\psi|$

when left bracket × right bracket:

$\langle\psi|\psi\rangle = \int \psi^* \cdot \psi \cdot dx$ (inner product).

∴ Expected energy: $\langle E\rangle = \langle\psi|H|\psi\rangle$.

Interestingly, I: unit vector, can be written as:

$I = \sum_n |\psi_n\rangle\langle\psi_n|$; where ψ_n is eigenstate of H.

Prove: $I|\psi\rangle = I \cdot \sum_m a_m |\psi_m\rangle = \sum_n |\psi_n\rangle\langle\psi_n|\, a_m \cdot |\psi_m\rangle$

$= \sum_n a_m |\psi_n\rangle \cdot \delta_{m,n} = \sum_m a_m |\psi_m\rangle = |\psi\rangle$.

$n = m$, non-zero.

☆ Orthognal Proof

Eigenfunction means to an operator A (Hermite operator)

$$A \cdot \psi_n = a_n \cdot \psi_n,$$ where a_n is a number.

If $a_n \neq a_m$, we can try calculate:

$$\langle \psi_n | A | \psi_m \rangle = a_m \langle \psi_n | \psi_m \rangle$$

or

$$\langle \psi_n | A | \psi_m \rangle = \langle A^{+} \psi_n | \psi_m \rangle = \langle A \psi_n | \psi_m \rangle$$

→ Hermitian

$$= a_n \langle \psi_n | \psi_m \rangle$$

$$\therefore \langle \psi_n | A | \psi_m \rangle = a_m \langle \psi_n | \psi_m \rangle = a_n \langle \psi_n | \psi_m \rangle$$

$\because a_n \neq a_m$, $\therefore \langle \psi_n | \psi_m \rangle = 0$.

When $a_n = a_m$, it's degenerate.

then $\psi' = C_1 \psi_n + C_2 \psi_m$ is still eigenfunction with eigenvalue $a_n (a_m)$

$$A \psi' = C_1 A \psi_n + C_2 A \psi_m = a_{n(m)} (C_1 \psi_n + C_2 \psi_m)$$

So we can always find a combination, where:

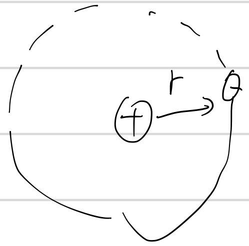
$$\psi_+ = C_+ \psi_n + C'_+ \psi_m \ ; \ \psi_- = C_- \psi_n + C'_- \psi_m$$

Such that $\int \psi_+^* \psi_- \, dx = 0$.

"Sidenote"

Check of De Broglie approach to Hydrogen Atom.

$$n \cdot \lambda = 2\pi \cdot r.$$



$$\frac{e^2}{4\pi\varepsilon_0 \cdot r^2} = \frac{mv^2}{r}$$

De Broglie wave assumption:

$$\hbar k = p = mv; \quad \hbar \cdot \frac{2\pi}{\lambda} = mv = \frac{h}{\lambda}$$

$$\lambda = \frac{2\pi r}{n} = \frac{h}{mv}; \quad mv = \frac{n \cdot h}{2\pi r} = \frac{n \cdot \hbar}{r}$$

This is consistant with Neil Bohr's model, where angular momentum is assumed to be quantized.

$$mvr = n\hbar.$$

And as we have: $\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r} = mv^2 \Rightarrow r = \frac{e^2}{4\pi\varepsilon_0 \cdot mv^2}$

$$\Rightarrow mvr = \frac{m \cdot v \cdot e^2}{4\pi\varepsilon_0\, m \cdot v^2} = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{v} = n\hbar \Rightarrow v = \frac{e^2}{4\pi\varepsilon_0\hbar} \cdot \frac{1}{n}.$$

$$E(r) = \frac{-e^2}{4\pi\varepsilon_0}\frac{1}{r} + \frac{1}{2}mv^2 = -\frac{1}{2}mv^2$$

$$= -\frac{me^4}{32\pi^2\varepsilon_0^2\hbar^2} \cdot \frac{1}{n^2}$$

So the energy difference between each state is:

$$\Delta E_{n_1, n_2} = \frac{me^4}{32\pi^2 \varepsilon_0^2 \hbar^2} \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right) \propto \left(\frac{1}{n_1^2} - \frac{1}{n_2^2} \right).$$

Correct.

Let's approach our two level system.

$|2\rangle$

$E\cos\omega t$

$|1\rangle$

$$H = H_0 - q\cdot\hat{x}\cdot\vec{E}(t)$$

H_0: energy of non-perturbed atom

$q\cdot\hat{x}\cdot\vec{E}(t)$: potential energy in $\vec{E}$ field. similar to gravity

And previously we already know: $q\cdot\hat{x}$, or $q\cdot\vec{r}$ is dipole.

Let: $q\cdot\hat{x} = \hat{\mu}$.

$\therefore\ H = H_0 - \hat{\mu}\cdot\vec{E}(t)$

We take $\hat{\mu}\cdot\vec{E}(t)$ relatively small compared to H_0, so that the eigenstate is still $|1\rangle, |2\rangle$; the eigenfrequency is E_1 and E_2.

Now what is H_0? H_0 could be complicated.
The simplest case is hydrogen atom, $H_0 = \frac{p^2}{2m} - \frac{e^2}{4\pi\varepsilon_0 r}$.

However, this form of H_0 is not helpful.

We want to solve $\psi(t) = a(t)\psi_1 + b(t)\psi_2$

$a(t)$ and $b(t)$ in;

we don't really care the actual form of $\psi_1(r)$, $\psi_2(r)$

So we fully make use of $|1\rangle$, $|2\rangle$;
we have:

$$H_0|1\rangle = E_1|1\rangle;$$ eigenstate:
$$H_0|2\rangle = E_2|2\rangle;$$

And $H_0(a|1\rangle + b|2\rangle) = aE_1|1\rangle + bE_2|2\rangle$;

If there's an algebra form of H_0 that can maintain these relations above, it is a valid representation of H_0.

$$H_0 = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2|$$

Verify: $H_0|1\rangle = E_1|1\rangle\underbrace{\langle 1|1\rangle}_{=1} + E_2|2\rangle\underbrace{\langle 2|1\rangle}_{=0}$

$= E_1|1\rangle$

$H_0|2\rangle = \cdots = E_2|2\rangle$;

Linear combination: $H_0(a|1\rangle + b|2\rangle) = aE_1|1\rangle + bE_2|2\rangle$;

So: $H = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2| - \vec{\mu}\cdot\vec{E}(t)$
$= E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2| - \vec{\mu}\cdot\vec{E}\cdot\cos\omega t.$

We want to solve: $|\psi(t)\rangle = a(t)|1\rangle + b(t)|2\rangle$;
from $i\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$.

Since we know $\langle 1|1\rangle = \langle 2|2\rangle = 1$; $\langle 1|2\rangle = \langle 2|1\rangle = 0$.

If we can express $\vec{\mu}\cdot\vec{E}\cos\omega t$ as $|1\rangle$, $|2\rangle$ term, then we have very high chance to solve it.

A standard procedure to project an operator (H) to a vector set ($|1\rangle$, $|2\rangle$).

We start with H_0 as an example:

$$H_0 = \hat{I} H_0 \hat{I} = \sum_{m,n} |m\rangle\langle m| H_0 |n\rangle\langle n| .$$

$\hat{I}$: unit vector

In our case: $= \left(|1\rangle\langle 1| + |2\rangle\langle 2|\right) H_0 \left(|1\rangle\langle 1| + |2\rangle\langle 2|\right)$

$$= \left(|1\rangle\langle 1| + |2\rangle\langle 2|\right)\left(E_1 |1\rangle\langle 1| + E_2 |2\rangle\langle 2|\right)$$

$$= E_1 |1\rangle\langle 1| + E_2 |1\rangle\underbrace{\langle 1|2\rangle}_{=0}\langle 2| + E_1 |2\rangle\underbrace{\langle 2|1\rangle}_{=0}\langle 1|$$

$$+ |2\rangle\langle 2| E_2 |2\rangle\langle 2|$$

$$= E_1 |1\rangle\langle 1| + E_2 |2\rangle\langle 2| .$$

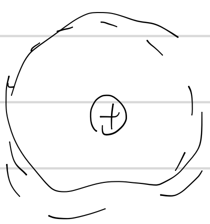
Similarly, we can do: $\vec{\mu}\cdot\vec{E}\cos\omega t$

$\hat{I}\, q\hat{x}\cdot\vec{E}\, \hat{I}$; $\vec{E}$ is a vector, not an operator, so doesn't matter.

$$q\hat{x}\vec{E} = \left(|1\rangle\langle 1| + |2\rangle\langle 2|\right) q\hat{x}\cdot\vec{E} \left(|1\rangle\langle 1| + |2\rangle\langle 2|\right)$$

$$= |1\rangle\langle 1| q\hat{x} |1\rangle\cdot\vec{E}\langle 1| + |1\rangle\langle 1| q\hat{x} |2\rangle\cdot\vec{E}\langle 2|$$
$$+ |2\rangle\langle 2| q\hat{x} |1\rangle\vec{E}\langle 1| + |2\rangle\langle 2| q\hat{x} |2\rangle\vec{E}\langle 2|$$

Now, we know the atoms have spherical symmetry



So $\psi_1(\vec{x})$; $\psi_2(\vec{x})$ are symmetry to ~~to~~ center:

$$\langle 1|\hat{x}|1\rangle = \int \psi_1(\vec{x})^* \vec{x}\, \psi_1(\vec{x})\, d\vec{x} \longrightarrow 0.$$

$$\langle 2|\hat{x}|2\rangle = 0$$

$$\therefore\ q\hat{x}\vec{E}(t) = |1\rangle\langle 1|q\hat{x}|2\rangle \vec{E}\langle 2| + |2\rangle\langle 2|q\hat{x}|1\rangle \vec{E}\langle 1|$$

Let's define: $\langle 1|q\hat{x}|2\rangle \equiv \vec{\mu}_{12}$; $\langle 2|q\hat{x}|1\rangle \equiv \vec{\mu}_{21}$

$$\vec{\mu}_{12} = \vec{\mu}_{21}^*$$

This dipole is a property of that atom, not related to external field.

$$\mu_{12} = q \cdot \int \psi_1^*(\vec{x}) \cdot \vec{x} \cdot \psi_2(\vec{x})\, dx\ ;$$

$$\therefore\ q\hat{x}\vec{E}(t) = \vec{\mu}_{12} \cdot E\cos\omega t \cdot |1\rangle\langle 2| + \vec{\mu}_{21} E\cos\omega t\ |2\rangle\langle 1|$$

$$\therefore\ H = H_0 - q\hat{x}\vec{E}\cos\omega t$$

$$= E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2| - \vec{\mu}_{12} E\cos\omega t\ |1\rangle\langle 2| + \vec{\mu}_{21} E\cos\omega t\ |2\rangle\langle 1|$$

One way to write this Hamiltonian is through matrix: the label 1, 2. is the same as column and row; Also the multiplication rules are the same.

$$H = \begin{pmatrix} E_1 & -\vec{\mu}_{12}\cdot\vec{E}\cos\omega t \\ -\vec{\mu}_{21}\cdot\vec{E}\cos\omega t, & E_2 \end{pmatrix}$$

Accordingly: $|\psi(t)\rangle = a|1\rangle + b|2\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

Verify that $H|\psi(t)\rangle$ equals to the same when doing matrix and algebra.

Now solution:

We know $H|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle$

For eigenstate, we have:

$H|1\rangle = E_1|1\rangle$; The only way to match Schrodinger Eq. is that:

$\psi_1(t) \rightarrow e^{-iE_1t/\hbar}|1\rangle$

$\therefore H\psi_1(t) = H\cdot e^{-iE_1t/\hbar}|1\rangle = e^{-iE_1t/\hbar}\cdot E_1|1\rangle$;

$i\hbar\frac{\partial}{\partial t}\psi_1(t) = i\hbar\frac{\partial}{\partial t}e^{-iE_1t/\hbar}|1\rangle = E_1 e^{-iE_1t/\hbar}\cdot|1\rangle$

$\therefore H\psi_1(t) = i\hbar\frac{\partial}{\partial t}\psi_1(t)$.

Same for $\psi_2(t) = e^{-iE_2t/\hbar}|2\rangle$.

So our solution would be:

$$\psi(t) = a(t)\, e^{-iE_1 t/\hbar}\, |1\rangle + b(t)\, e^{-iE_2 t/\hbar}\, |2\rangle$$

$$H\psi(t) = i\hbar \frac{\partial}{\partial t}\psi(t);$$

$$H = E_1 |1\rangle\langle 1| + E_2 |2\rangle\langle 2| - \vec{\mu}_{12}\cdot\vec{E}\cos\omega t\, |1\rangle\langle 2| - \vec{\mu}_{21}\cdot\vec{E}\cos\omega t\, |2\rangle\langle 1|$$

plug in:

$$H|\psi(t)\rangle = E_1 e^{-iE_1 t/\hbar} a(t)\, |1\rangle - \vec{\mu}_{21}\cdot\vec{E}\cos\omega t \cdot e^{-iE_1 t/\hbar} a(t)\, |2\rangle$$
$$+ E_2 e^{-iE_2 t/\hbar} b(t)\, |2\rangle - \vec{\mu}_{12}\cdot\vec{E}\cos\omega t\cdot e^{-iE_2 t/\hbar} b(t)\, |1\rangle$$

$$= i\hbar\frac{\partial}{\partial t}\psi(t) = E_1 e^{iE_1 t/\hbar} a(t)|1\rangle + i\hbar e^{-iE_1 t/\hbar}\dot{a}(t)\, |1\rangle$$
$$+ E_2 e^{iE_2 t/\hbar} b(t)|2\rangle + i\hbar e^{-iE_2 t/\hbar}\dot{b}(t)\, |2\rangle$$

Cancel terms, we get:

$$-\vec{\mu}_{12}\vec{E}\cos\omega t\, e^{-iE_2 t/\hbar} b(t)\, |1\rangle - \vec{\mu}_{21}\vec{E}\cos\omega t\, e^{-iE_1 t/\hbar} a(t)\, |2\rangle$$

$$= i\hbar e^{-iE_1 t/\hbar}\dot{a}(t)|1\rangle + i\hbar\, e^{-iE_2 t/\hbar}\dot{b}(t)\, |2\rangle$$

Apply $\langle 1|$ to the equation, we get:

$$\dot{a}(t) = \frac{i\vec{\mu}_{12}\cdot\vec{E}}{\hbar}\cos\omega t\, e^{-i(E_2-E_1)t/\hbar}\, b(t)$$

$$\dot{b}(t) = \frac{i\vec{\mu}_{21}\cdot\vec{E}}{\hbar}\cos\omega t\, e^{-i(E_1-E_2)t/\hbar}\, a(t)$$

Let: $E_2 - E_1 = \hbar\Omega$; we have:

$$\boxed{\begin{aligned} \dot{a}(t) &= \frac{i\vec{\mu}_{12}\cdot\vec{E}}{\hbar}\cos\omega t\, e^{-i\Omega t}\, b(t) \\ \dot{b}(t) &= \frac{i\vec{\mu}_{21}\cdot\vec{E}}{\hbar}\cos\omega t\, e^{i\Omega t}\, a(t) \end{aligned}}$$

To further continue the calculation, we can expand:

$$\cos\omega t = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$$

$$\Rightarrow \dot{a}(t) = \frac{i\vec{\mu}_{12}\cdot\vec{E}}{2\hbar}\left(e^{i(\omega-\Omega)t} + e^{-i(\omega+\Omega)t}\right) b(t)$$

$$\dot{b}(t) = \frac{i\vec{\mu}_{21}\cdot\vec{E}}{2\hbar}\left(e^{i(\omega+\Omega)t} + e^{-i(\omega-\Omega)t}\right) a(t).$$

Rotation wave approximation:

If we write $a(t)$ in intergration form:

$$a(t) = \frac{i\vec{\mu}_{12}\cdot\vec{E}}{2\hbar}\int e^{+i(\omega-\Omega)t}\, b(t) + \int e^{-i(\omega+\Omega)t}\, b(t).$$

In optics domain, $\omega - \Omega \sim$ GHz (linewidth of resonance)

$\omega + \Omega \rightarrow$ 100s THz (light frequency)

So $e^{-i(\omega+\Omega)t}$ is much much faster than $e^{+i(\omega-\Omega)t}$

So when we consider $a(t)$ at the time scale of $t \sim \frac{1}{\omega-\Omega}$, we can take the much faster $e^{i(\omega+\Omega)t}$ to its average value. $\int e^{i(\omega+\Omega)t} b(t)\, dt = b(t) \int e^{i(\omega+\Omega)t} dt \to 0.$

assume much slower than $e^{i(\omega+\Omega)t}$

$$\boxed{\begin{aligned} \text{So:} \quad \dot{a}(t) &= \frac{i \vec{\mu}_{12} \cdot \vec{E}}{2\hbar} e^{i(\omega-\Omega)t} b(t) \\ \dot{b}(t) &= \frac{i \vec{\mu}_{21} \cdot \vec{E}}{2\hbar} e^{-i(\omega-\Omega)t} a(t) \end{aligned}}$$

Further tip on Rotation wave approximation:

In full quantum picture, we can get

$e^{-i(\omega+\Omega)t}$ Correspond to a process where a photon is absorped, and atom moves from $|2\rangle \to |1\rangle$.

It violates energy conservation.

Definition: Let $\omega - \Omega = \delta\omega$, is detuning.

Assume the simplest case, $\vec{\mu}_{12}$ is real. so $\vec{\mu}_{21} = \vec{\mu}_{12}^* = \vec{\mu}_{12}$

And: $\Omega_R = \frac{\vec{\mu}_{12} \cdot \vec{E}}{\hbar}$: Rabi Frequency

describe the coupling strength between $|1\rangle$ and $|2\rangle$

$$\therefore \quad \dot{a}(t) = \frac{i\Omega_R}{2} b(t) e^{-i\delta\omega t}$$

$$\dot{b}(t) = \frac{i\Omega_R}{2} a(t) e^{i\delta\omega t}$$

Again, further simplify by assuming $\delta\omega = 0 \Rightarrow \omega = \Omega$.

On resonance condition:

$$\begin{cases} \dot{a}(t) = \frac{i\Omega_R}{2} b(t) \\ \dot{b}(t) = \frac{i\Omega_R}{2} a(t) \end{cases}$$

This is standard oscillation equation, as
$a(t)$ convert to $b(t)$, and $b(t)$ convert back to $a(t)$

Take derivative of time on $\dot{a}(t)$ equation:

$$\ddot{a}(t) = \frac{i\Omega_R}{2} \dot{b}(t) = \frac{i\Omega_R}{2} \left(\frac{i\Omega_R}{2} a(t) \right) = -\frac{\Omega_R^2}{4} a(t)$$

$$\Rightarrow \ddot{a}(t) + \frac{\Omega_R^2}{4} a(t) = 0$$

Universal solution: $a(t) = C_1 \cos\left(\frac{\Omega_R}{2} t\right) + C_2 \sin\left(\frac{\Omega_R}{2} t\right)$

As $\dot{a}(t) = \frac{i\Omega_R}{2} b(t)$,

$$b(t) = \frac{-2i}{\Omega_R} \times \frac{\Omega_R}{2} \left(C_2 \cos\left(\frac{\Omega_R}{2} t\right) - C_1 \sin\left(\frac{\Omega_R}{2} t\right) \right)$$

$$= -i \left(C_2 \cos\frac{\Omega_R}{2} t - C_1 \sin\frac{\Omega_R t}{2} \right)$$

Initial condition: at $t=0$, atom in ground state.

So: $a(0)=1$; $b(0)=0$;

$b(0)=0 \rightarrow C_2=0$;

$a(0)=1 \rightarrow C_1=1$;

$\therefore\ a(t) = \cos\left(\frac{\Omega_R}{2}t\right)$; $b(t) = i\sin\left(\frac{\Omega_R t}{2}\right)$

$$\therefore\ \psi(t) = a(t)e^{-iE_1t/\hbar}|1\rangle + b(t)e^{-iE_2t/\hbar}|2\rangle$$

$$= e^{-iE_1t/\hbar}\left(a(t)|1\rangle + b(t)e^{-i\Omega t/2}|2\rangle\right).$$

Probability in $|1\rangle$ is: $|a(t)|^2$

$$P_1(t) = |a(t)|^2 = \cos^2\left(\frac{\Omega_R}{2}t\right) = \frac{1}{2}(1+\cos\Omega_R t)$$

$$P_2(t) = |b(t)|^2 = \sin^2\left(\frac{\Omega_R}{2}t\right) = \frac{1}{2}(1-\cos\Omega_R t)$$