Sidenote 1. Maxwell equation

① Gauss's Law

comes from Coulomb's Law

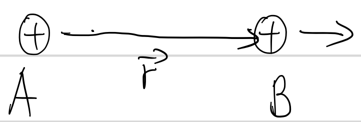
Gauss's Law: $\nabla \cdot \vec{E} = \frac{\rho_{free}}{\varepsilon}$; $\varepsilon = \varepsilon_0 \cdot \varepsilon_r$,

ε_0 is the vacuum permittivity, ε_r is the relative permittivity.

$\nabla \cdot (\ldots)$ is the divergence of $(\ldots)$

$$\nabla \cdot \vec{E} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (E_x, E_y, E_z)$$

Coulomb's Law (from experiment)

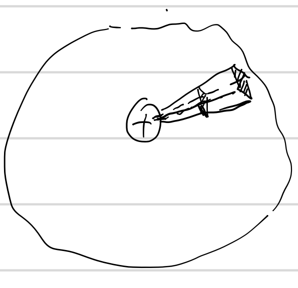


$$\vec{F}_B = \frac{q_A \cdot q_B}{4\pi\varepsilon_0} \cdot \frac{\hat{e}_r}{r^2} = \frac{q_A q_B}{4\pi\varepsilon_0} \frac{\vec{r}}{|\vec{r}|^3}$$

We define field: $\vec{F}_B = q_B \cdot \vec{E}$; $\vec{E} = \frac{q_A \hat{e}_r}{4\pi\varepsilon_0 \cdot r^2}$

$\longrightarrow \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ go to this form.

Integration form of Coulomb's Law:



$\int \vec{E} \cdot ds$: Electric Flux.

$ds = r^2 \cdot d\theta$; $\vec{E} \propto \frac{1}{r^2}$;

$$\oiint_{\Sigma S} \vec{E} \cdot d\vec{S} = \iint \frac{q}{4\pi\varepsilon_0} \frac{1}{r^2} \cdot r^2 \cdot d\theta \cdot d\phi = \frac{q}{4\pi\varepsilon_0} \cdot 4\pi = \frac{q}{\varepsilon_0}.$$

Intergration form of Coulomb's Law, Guass's Law:

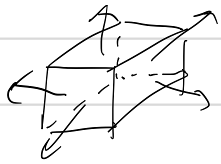
$$\oiint_{\Sigma S} \vec{E} \cdot d\vec{s} = \frac{q}{\varepsilon_0}.$$

This is true for charges, because $\vec{F} = \vec{F_A} + \vec{F_B} + \vec{F_C} + \dots$

$$\oiint_{\Sigma S} \vec{E} \cdot d\vec{s} = \Sigma \frac{q}{\varepsilon_0} = \oiiint_{\Sigma V} \frac{\rho}{\varepsilon_0} dV \; ; \quad \rho: \text{charge density}$$

Divergence Theorem: (Guass's Law)

Conservation Law:



$$\oiint_{\Sigma S} \vec{F} \cdot \hat{n} \, ds$$

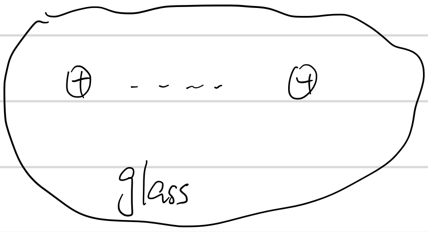
$$= \oiiint_{\Sigma V} (\nabla \cdot \vec{F}) \, dV$$

Combine:

$$\oiiint_{\Sigma V} (\nabla \cdot \vec{E}) \, dV = \oiiint_{\Sigma V} \frac{\rho}{\varepsilon_0} dV, \quad \text{hold for all } \Sigma V.$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}.$$

This is the case in vacuum, what about in dielectric?



$$\vec{F} = \frac{1}{4\pi \varepsilon_r \varepsilon_0} \frac{q_A q_B \hat{e}_r}{r^2}$$

$n = 1.45.$ ε_r: relative permittivity.

$$\nabla \cdot \vec{E} = \frac{\rho_{free}}{\varepsilon_r \varepsilon_0} = \frac{\rho_{free}}{\varepsilon}. \quad ; \quad \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

Difficulties: ①: For $\nabla \cdot \vec{E} = \frac{\rho_{free}}{\varepsilon}$;

You limit the possibility of the material response; $\varepsilon = \varepsilon(\vec{E}, \dots)$

②. $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$; Because you need to calculate ρ!

$$\rho = \rho_{free} + \rho_{bound}.$$

Maxwell defines: $\nabla \cdot \vec{D} = \rho_{free}$; $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$.

$\vec{P}$: polarization.

In linear regime, we know: $\nabla \cdot \vec{E} = \frac{\rho_{free}}{\varepsilon}$

$\Rightarrow \varepsilon_r \varepsilon_0 \vec{E} = \vec{D} = \varepsilon_0 \vec{E} + \vec{P}$;

$\vec{P} = (\varepsilon_r - 1) \varepsilon_0 \vec{E} = \chi \varepsilon_0 \vec{E}$.

$$\nabla \cdot \vec{D} = \varepsilon_0 \nabla \cdot \vec{E} + \nabla \cdot \vec{P} = \rho_{free}$$

And we know: $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$

$$\therefore \quad \nabla \cdot \vec{P} = \rho_{free} - \rho_{tot}.$$

$\vec{P}$ is called induced electric dipole moment density, polarization.

$\vec{P}$ describe the electric response of the material.

$\vec{P}$ is the most important thing in our class, and we will derive how it back action to $\vec{E}$.

$$\vec{P} = (\varepsilon_r - 1)\varepsilon_0 \vec{E} + \vec{P}_{NL}$$

$$\nabla \cdot \vec{D} = \rho_{free} \; ; \quad \vec{D} = \varepsilon_0 \vec{E} + \vec{P} \; ; \quad \vec{P} = (\varepsilon_r - 1)\varepsilon_0 \vec{E} + \vec{P}_{NL} .$$

Guass's Law, Coulomb's Law.

② Guass's Law for magnetism. (No magnetic monopoles)

$$\nabla \cdot \vec{B} = 0.$$

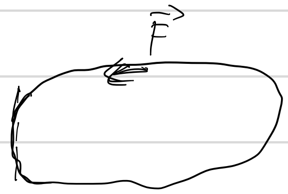
③ Maxwell - Faraday equation

Faraday's Law of induction.

The induced electromotive force in any closed circuit is equal to the negative of the time rate of change of the magnetic flux enclosed by the circuit.

$$\vec{\mathcal{E}} = -\frac{d}{dt}\Phi_B \quad ; \quad \Phi_B = \oiint_{\Sigma} \vec{B} \cdot d\vec{s}.$$

$$\oint_{\Sigma l} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \oiint_{\Sigma} \vec{B} \cdot d\vec{s}.$$

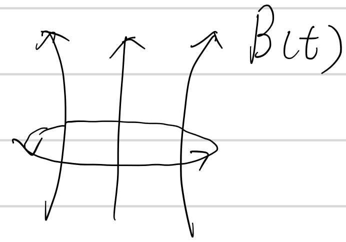


Stokes Theorem (Curl theorem)

$$\oint_{\Sigma l} \vec{F} \cdot d\vec{l} = \oiint_{S} (\nabla \times \vec{F}) \cdot ds$$

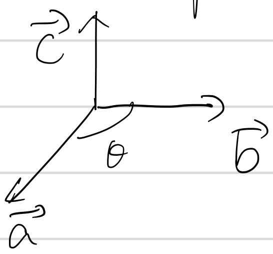
$$\Rightarrow \oiint_{S} (\nabla \times \vec{E}) \cdot d\vec{s} = -\frac{d}{dt} \oiint_{S} \vec{B} \cdot d\vec{s}$$

$$\Rightarrow \nabla \times \vec{E} = -\frac{d}{dt}\vec{B}$$



Key to EM wave

Cross product: $\vec{a} \times \vec{b} = \vec{c}$.

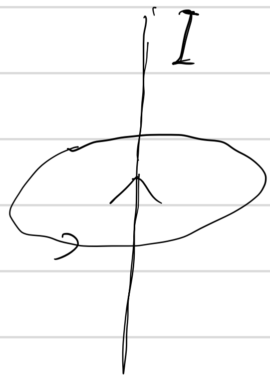


$|\vec{c}| = |a| \cdot |b| \cdot \sin\theta$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$ Determinant

$$= \hat{e}_x (a_y b_z - a_z b_y) + \hat{e}_y (a_z b_x - b_z a_x) + \hat{e}_z (a_x b_y - a_y b_x)$$

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$



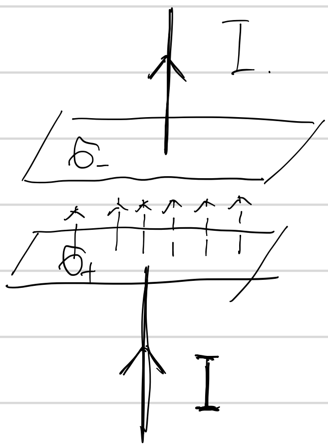
④. Ampère's circuital Law

Original form is: $\oint_{\Sigma l} \vec{B} \cdot d\vec{l} = \Sigma \mu_0 I$

$$\oint_{\Sigma l} \vec{B} \cdot d\vec{l} = \mu_0 \oiint_{\Sigma s} \vec{J} \cdot d\vec{s}.$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}.$$

Thought experiment.



capacity

$\sigma_{+(-)}$: surface density of charge.

$\sigma_{+(-)} \nearrow$ with time.

$\vec{E}$ or $\vec{D}$? Free charge. $\nabla \cdot \vec{D} = \rho_{free}$

$$\vec{D}_+ = \frac{\sigma_+}{2} \quad ; \quad \vec{D}_- = \frac{\sigma_-}{2}$$

$$\vec{D} = \sigma \quad ; \quad \frac{\partial \sigma}{\partial t} \rightarrow I.$$

$\frac{\partial \vec{D}}{\partial t} \longrightarrow$ rate of charge change $\rightarrow I.$

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right).$$

$$\begin{cases} \nabla \cdot \vec{D} = \rho_{free} \,;\; \vec{D} = \varepsilon_0 \vec{E} + \vec{P} \,;\; \vec{D} = \varepsilon_0 \varepsilon_r \vec{E} + \vec{P}_{NL}. \\ \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{B} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \end{cases}$$